Slovak University of Technology in Bratislava
Faculty of Electrical Engineering and Information Technology
Institute of Robotics and Cybernetics

Doctoral Thesis

Gain-Scheduled Controller Design

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Slovenská technická univerzita v Bratislave
Ústav robotiky a kybernetiky

Fakulta elektrotechniky a informatiky

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Miesto vypracovania:	ÚRK
Názov práce:	**Riadenie systémov metódou "gain scheduling"**
Špecifikácia zadania:	Špecifikácia zadania: Dizertačná práca bude venovaná problematike návrhu regulátora s plánovaným zosilnením (gain-scheduled). Cieľom práce je nájsť systematický postup na návrh optimálnych (suboptimálnych) regulátorov s plánovaným zosilnením pri obmedzení vstupno/výstupných hodnôt systémov. Návrh realizujte aj pre nelineárne systémy s neurčitosťami.
Dátum zadania:	24. 08. 2012
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"You know that children are growing up when they start asking questions that have answers."

John J. Plomp

"You know that children are growing up when they start asking questions that have answers."

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Abstract

(English)

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Doctor of Philosophy

Gain-Scheduled Controller Design

by Adrian Ilka, Ing.

This thesis is devoted to controller synthesis, i.e. finding a systematic procedure to determine the optimal (sub-optimal) controller parameters which guarantee the closed-loop stability and guaranteed cost for uncertain nonlinear systems with considering input/output constraints, all this without on-line optimization. The controller in this thesis is given in a feedback structure that is, the controller has information about the system and uses this information to influence the system. In this thesis the linear parameter-varying based gain scheduling is investigated. The nonlinear system is transformed to a linear parameter-varying system, which is used for controller design, i.e. a gain-scheduled controller design with consideration of the objectives on the system. The gain-scheduled controller synthesis in this thesis is based on the Lyapunov theory of stability as well as on the Bellman-Lyapunov function. Several forms of parameter dependent/quadratic Lyapunov functions are presented and tested. To achieve performance quality, a quadratic cost function and its modifications known from LQ theory are used. In this thesis one can also find an application of gain scheduling in switched and in model predictive control with consideration of input/output constraints. The main results for controller synthesis are in the form of bilinear matrix inequalities (BMI) and/or linear matrix inequalities (LMI). For controller synthesis one can use a free and open source BMI solver PenLab or LMI solvers LMILab or SeDuMi. The synthesis can be done in a computationally tractable and systematic way, therefore the linear parameter-varying based gain scheduling approach presented in this thesis is a worthy competitor to other controller synthesis methods for nonlinear systems.

Keywords: *Gain-scheduled control; Lyapunov theory of stability; Guaranteed cost control; Bellman-Lyapunov function; LPV system; Robust control; Input/output constraints*

Abstrakt

(Slovak)

SLOVENSKÁ TECHNICKÁ UNIVERZITA V BRATISLAVE
Fakulta elektrotechniky a informatiky
Ústav robotiky a kybernetiky

Doctor of Philosophy

Riadenie systémov metódou "gain scheduling"

Adrian ILKA, Ing.

Táto práca sa venuje problematike návrhu regulátora, tj. nájsť systematický postup na návrh optimálnych (suboptimálnych) parametrov regulátora, ktoré garantujú stabilitu a kvalitu v uzavretej slučke, pri obmedzení vstupno-výstupných hodnôt systémov pre nelineárne systémy s neurčitosťami, a to bez on-line optimalizácie. Uvedený regulátor má spätno-väzobnú riadiacu štruktúru, čo znamená, že disponuje informáciami o danom systéme, ktoré využíva k jeho ovplyvneniu. Táto práca sa podrobnejšie zaoberá s riadením s plánovaným zosilnením, a to na báze parametricky závislých lineárnych systémov. Nelineárny systém je pretransformovaný na parametricky závislý lineárny systém, čo sa následne využíva na návrh regulátora, tj. regulátora s plánovaným zosilnením, s ohľadom na požiadavky daného systému. Syntéza regulátora s plánovaným zosilnením sa uskutoční na báze Lyapunovej teórie stability s použitím Bellman-Lyapunovej funkcie, v rámci čoho sú prezentované a testované rôzne typy kvadratickej a parametricky závislej Lyapunovej funkcie. Pre dosiahnutie požadovanej kvality sa používa kvadratická účelová funkcia známa z LQ riadenia, s rôznymi modifikáciami. V tejto práci nájdeme aj aplikáciu riadenia s plánovaným zosilnením v oblasti takzvaného prepínacieho riadenia (switched control), ako aj v rámci prediktívneho riadenia (model predictive control). Hlavné výsledky pre syntézu regulátorov sú v tvare bilineárnych maticových nerovníc (BMI) a/alebo linearných maticových nerovníc (LMI). Na návrh regulátorov môžeme používať bezplatný a „open source" BMI solver PenLab alebo LMI solvre LMILab a SeDuMi. Uvedené skutočnosti umožnia vykonať syntézu jednoduchým a systematickým spôsobom. Riadenie s plánovaným zosilnením na báze parametricky závislých lineárnych systémov prezentované v tejto práci je vhodným konkurentom vo vzťahu k iným metódam syntézy regulátorov pre nelineárne systémy.

Kľúčové slová: *Riadenie s plánovaným zosilnením; Lyapunová teória stability; Riadenie s garantovanou kvalitou; Bellman-Lyapunová funkcia; LPV systémy; Robustné riadenie; Vstupné/výstupné obmedzenia*

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Abbreviations

A/D	**A**nalog / **D**igital
AQS	**A**ffine **Q**uadratic **S**stability
BMI	**B**ilinear **M**atrix **I**nequality
BW	**B**ody **W**eight
CGM	**C**ontinuous **G**lucose **M**onitoring
DPC	**D**ynamic **P**roperty **C**oefficient
GS	**G**ain-**S**cheduling
GSC	**G**ain-**S**cheduled **C**ontroller
GSLMIDP	**G**ain-**S**cheduled **L**inear **M**atrix **I**nequality **D**esign **P**rocedure
LFT	**L**inear **F**ractional **T**ransformation
LMI	**L**inear **M**atrix **I**nequality
LPV	**L**inear **P**arameter-**V**arying
LQ	**L**inear-**Q**uadratic
LTI	**L**inear **T**ime **I**nvariant
MIMO	**M**ulti **I**nput **M**ulti **O**utput
MPC	**M**odel **P**redictive **C**ontrol
MPQS	**M**ulti **P**arameter **Q**uadratic **S**tability
MU	**M**achine **U**nit
NCSs	**N**etworked **C**ontrol **S**ystems
PDQS	**P**arameter **D**ependent **Q**uadratic **S**tability
PID	**P**roportional-**I**ntegral-**D**erivative
PK	**P**harmaco**K**inetics
SISO	**S**ingle **I**nput **S**ingle **O**utput
T1DM	**T**ype **O**ne **D**iabetes **M**ellitus
WP	**W**orking **P**oint

To my wonderful wife Viktória

1

Introduction

This thesis is devoted to controller synthesis, i.e. finding a systematic procedure to determine the optimal (sub-optimal) controller parameters which guarantee the closed-loop stability and guaranteed cost for uncertain nonlinear systems with considering input/output constraints. In consideration of the objectives stated for the system such as tracking a reference signal or keeping the plant at a desired working point (operation point) and based on the knowledge of the system (plant), the controller takes decisions. In this thesis, the controller is given in a feedback structure, which means that the controller has information about the system and uses it to influence the system. A system with a feedback controller is said to be a closed-loop system.

To design a controller which satisfies the objectives, we need an adequately accurate model of the physical system. Nevertheless, real plants are hard to describe exactly. Alternatively, the designed controller must handle cases when the state of the real plant differs from what is observed by the model. A controller that is able to handle model uncertainties and/or disturbances is said to be robust, and the theory dealing with these issues is said to be robust control.

The robust control theory is well established for linear systems but almost all real processes are more or less nonlinear. If the plant operating region is small, one can use robust control approaches to design a linear robust controller, where the nonlinearities are treated as model uncertainties. However, for real nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable because the linear robust controller may not be able to meet the performance specifications. For this reason, the controller design for nonlinear systems is nowadays a very determinative and important field of research.

Gain scheduling is one of the most commonly used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. Many of the early articles were associated with flight control and aerospace. Then, gradually, this approach has been used almost everywhere in control engineering, which was greatly advanced with the introduction of LPV systems.

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. These were introduced first by Jeff S. Shamma in 1988 to model gain scheduling. Today the LPV paradigm has become a standard formalism in the area of systems and controls with lot of contributions and articles devoted to analysis, controller design and system identification of these models.

This thesis deals with linear parameter-varying based gain scheduling, which means that the nonlinear system is transformed to a linear parameter-varying system, which is used to design a controller, i.e. a gain-scheduled controller. The problem formulation is close to the linear system counterpart, therefore using LPV models for controller design has potential computational advantages over other controller synthesis methods for nonlinear systems. Not to mention that the LPV based gain scheduling approaches comes with a theoretical validity because the closed-loop system can meet certain specifications. Nonetheless, following the literature it is ascertainable that there are still many unsolved problems. This thesis is devoted to some of these problems.

1.1 Goals & Objectives

As already mentioned, there are many unsolved problems. Therefore, it is necessary to find new and novel controller design approaches. The main goal of this thesis is to find a controller design approach for uncertain nonlinear systems, which guarantees the closed-loop stability and the optimal controller output with considering input/output constraints, all this without on-line optimization and need of high-performance industrial computers. In order to achieve the above mentioned goal, we have set the following objectives:

- To suggest a gain-scheduled PID controller design approach with guaranteed cost in continuous and discrete time state space using BMI
- To suggest a robust gain-scheduled PID controller design approach with guaranteed cost and parameter dependent quadratic stability in state space using BMI
- To suggest a variable weighting gain-scheduled approach
- To convert some BMI controller design approaches to LMI
- To suggest a switched and model predictive gain-scheduled method
- To suggest a gain-scheduled controller design approach with input/output constraints
- To apply methods to relevant processes

1.2 Outline

The sequel of this thesis is organized as follows. In the preliminary chapter (*Chapter 2*), one can find a literature review with a brief overview of linear parameter-varying systems and gain scheduling. *Chapter 3* presents an overview of research results with a brief summary of included papers. After this, one can find 9 papers, which cover the main research results obtained within the last 2.5 years (*Chapter 4-12*). Finally, in *Chapter 13*, following the papers, some concluding remarks and suggestions for future research are given.

2

Preliminary chapter

In this chapter preliminaries of linear parameter-varying systems as well as gain scheduling are introduced. This chapter is intended to highlight the properties and give a short background to the tools used in the appended papers.

2.1 Linear parameter-varying systems

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. It was introduced first by Jeff S. Shamma in 1988 [1] to model gain scheduling. *"Today LPV paradigm has become a standard formalism in systems and controls with lot of researches and articles devoted to analysis, controller design and system identification of these models"*, as Shamma wrote in [2]. This section deals with LPV models and presents analytical approaches for LPV systems.

2.1.1 Introduction to LPV systems

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. Linear parameter-varying (LPV) systems have the following interpretations:

- they can be viewed as linear time invariant (LTI) plants subject to time-varying known parameters $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle$,
- they can be models of linear time-varying plants,
- they can be LTI plant models resulting from linearization of the nonlinear plants along trajectories of the parameter $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle$ which can be measured.

For the first and third class of systems, parameter θ can be exploited for the control strategy to increase the performance of closed-loop systems. Hence, in this thesis the following LPV system will be used:

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u \\ y &= Cx \end{aligned} \tag{2.1}$$

where for the affine case

$$\begin{aligned} A(\theta(t)) &= A_0 + A_1\theta_1(t) + \ldots + A_p\theta_p(t) \\ B(\theta(t)) &= B_0 + B_1\theta_1(t) + \ldots + B_p\theta_p(t) \end{aligned}$$

and $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control input, $y \in \mathbb{R}^l$ is the measurement output vector, A_0, B_0, A_i, B_i, $i = 1, 2 \ldots, p$, C are constant matrices of appropriate dimension, $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle \in \Omega$ and $\dot{\theta}(t) \in \langle \underline{\dot{\theta}}\ \overline{\dot{\theta}} \rangle \in \Omega_t$ are vectors of time-varying plant parameters which belong to the known boundaries.

The LPV paradigm was introduced by Jeff. S. Shamma in his Ph.D. thesis [1] for the analysis of gain-scheduled controller design. The authors in early works (see [1, 3–8] and surveys [9, 10]) in gain scheduling the LPV system framework called as the golden mean between linear and nonlinear dynamics, because "*the LPV system is an indexed collection of linear systems, in which the indexing parameter is exogenous, i.e., independent of the state.*"(wrote J. S. Shamma in his Ph.D. thesis [1]). In gain scheduling, this parameter is often a function of the state, and hence endogenous

$$\begin{aligned} \dot{x} &= A(z)x + B(z)u \\ y &= C(z)x \\ z &= h(x) \end{aligned} \tag{2.2}$$

2.1.1.1 Application of the LPV systems

Since the first publication devoted to LPV systems, the LPV paradigm has been used in several fields in control engineering including the modeling and control design. Traditionally the gain scheduling was the primary design approach for flight control and consequently many of the first articles and papers which applied and improved the LPV framework were associated with flight control. Afterwards continuously many papers and articles have appeared which are using LPV paradigm in several application areas such as:

- Flight control and missile autopilots [11–17]
- Aeroelasticity [18–21]
- Magnetic bearings [22–25]

- Automotive bearings [26–28]
- Energy and power systems [29–34]
- Turbofan engines [35–38]
- Microgravity [39–41]
- Diabetes control [42–44]
- Anesthesia delivery [45]
- IC manufacturing [46]
- etc.

Due to the success of LPV paradigm in 2012 for the twentieth anniversary of the invention of LPV paradigm a gift edition book was published by Javad Mohammadpour and Carsten W. Scherer Editors at Springer [2] which is fully devoted to LPV systems.

2.1.2 Stability analysis

The basic stability analysis question for LPV systems is how to ensure the stability of the closed-loop nonlinear system and of the closed-loop family of linear systems, when the scheduled parameters are changed. The following section is devoted to this basic stability question and shows the basic theoretical approaches to investigate the stability for

1. slow time parameter variations,
2. arbitrarily fast time parameter variations.

2.1.2.1 Time variations and instability

It is a well-known problem from linear system analysis that time variations can induce instability. For example, consider a stable LTV system (2.3), so the eigenvalues of $A(t)$ are in the left half plane for all $t \neq 0$. The question is for which solution the state $x(t)$ grows exponentially.

$$\dot{x} = A(t)x \tag{2.3}$$

Fig. 2.1 shows the main insight into this problem using the state trajectories of the LPV system (2.4) with parameter θ which is periodically switching between two values $\theta(t) \in \langle \omega_a, \omega_b \rangle$. In this figure the red line indicates the unstable switching trajectory and the *dashed lines* indicate individual oscillatory trajectories.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\theta^2 & 0 \end{pmatrix} x \tag{2.4}$$

For a fixed value of θ the LTI system is marginally stable. Instability occurs by an alignment of phases of increasing magnitude.

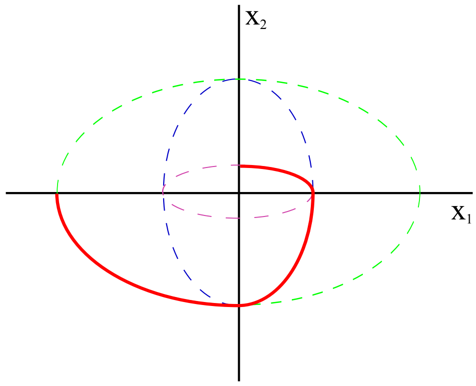

FIGURE 2.1: Instability induced by switching dynamics

Concerning to induced instability *non-minimum phasedness* is induced. The right-half-plane zeros in the transfer function of an LTI system can cause radical limitations in achievable performance. While time-varying systems do not have right-half-plane zeros, there are similar notions and similar resulting limits of performance. Shamma [2] defines a non-minimum phased property for nonlinear time-varying systems, where an unbounded input produces a bounded output. This property produces fundamental limitations on the closed-loop disturbance rejection. As Shmamma presented in [1], parameter time variation can induce instability, they can also induce such non-minimum phased behaviours. Summarizing all of this, an LPV system can be the minimum phase for constant parameter values, but non-minimum phase under time variation and thereby have fundamental limits on achievable performance that are not apparent from the constant parameter analysis.

2.1.2.2 Slow time parameter variations

In [2], Shamma has stated the following: "*Stability for constant parameter-varying parameter trajectories implies stability for slowly time-varying parameter trajectories*". This section presents a collection of results which motivated Shamma to formalize the previous statement.

Let Θ denote the set of admissible parameter values whereas $\mathcal{Q}$ denotes admissible trajectories for $\theta(\cdot)$, the related Θ denotes admissible values of $\theta(t)$. Let assume that for any θ_0, the LTI system is exponentially stable.

According to Shamma, in particular, let $m \geq 1$ and $\lambda > 0$ be such that for any $\theta_0 \in \Theta$, solution of (2.3) satisfy

$$|x(t)| \leq m e^{-\lambda t} |x(0)|$$

where m is referred to as a *peaking constant* which reflects that the state may increase in magnitude before decaying exponentially. *Fig. 2.2* shows the main principle of stability under slow time variations, where the red line indicates the actual state magnitude, the blue line indicates a succession of upper bounds implied by $me^{-\lambda t}$ and the green line is an exponentially decaying overall upper bound. For more details see [47].

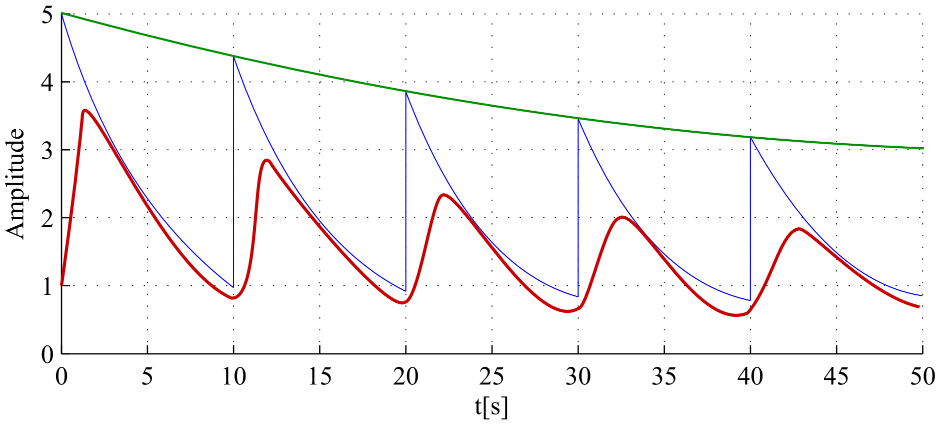

FIGURE 2.2: Stability and peaking

The statement *"Slow time-varying"* for the continuous case can be characterized as follows:

Assume Lipschitz continuity of $A(\cdot)$ for some $L_A > 0$

$$\|A(\theta) - A(\theta')\| \leq L_A|\theta - \theta'| \tag{2.5}$$

for all $\theta, \theta' \in \theta_c$. The expression $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$ and $\|A\|$ denotes the induced matrix norm. Than

- Persistently slow:
$$|\dot{\theta}| < \epsilon$$
- Slow on average:
$$\inf_{T>0} \sup_{t_0 \geq 0} \frac{1}{T} \int_{t_0}^{t_0+T} |\dot{\theta}|dt < \epsilon$$
over any interval $[t_0,\ t_0 + T]$ is small.

Theorem 2.1. *For all of the above settings the LPV system* (2.1) *is exponentially stable for a sufficiently small* $\epsilon > 0$.

Stability results for properly slow time variations, trace back to classical results in ordinary differential equations [48]. Nevertheless, a suitable analysis can derive revealing explicit bounds in the above case

- Persistently slow [1] and slow on average [49]:
$$\epsilon < \frac{\lambda^2}{4L_A m \log(m)}$$

Shamma stated an interesting implication from the above bounds, time variations can be arbitrarily fast, when $m = 1$. In terms of the previous discussion, $m = 1$ implies that trajectories in the constant parameter case have no peaking, and therefore cannot align to produce instability.

2.1.2.3 Arbitrary time parameter variations

This section deals with the stability question from the other extreme, when time variations are arbitrarily fast. For this discussion consider an LPV plant in the form

$$\dot{x}(t) = A(\theta(t))x(t) \tag{2.6}$$

Shamma and others [50–52] concluded that

- Determining whether solutions of (2.6) are bounded is undecidable.
- Determining whether (2.6) is asymptotically stable is NP-hard [1]
- Consequentially, deriving efficient algorithms for assessing stability will remain to be elusive.

Shamma according that a consequence of the complexity results is, that one must settle for non-definitive methods or inefficient algorithms to access stability.

Theorem 2.2. *The LPV system (2.1) is exponentially stable for all $\theta \in \Omega$ if there exist symmetric, positive defined matrix P such that the following inequality holds*

$$A^T(\theta)P + PA(\theta) < 0 \tag{2.7}$$

The proof is, that $x^T Px$ is a Lyapunov function for the LPV system, which in this case is the only Lyapunov function for all associated constant parameter LTI system. The result is only a sufficient condition. Shamma in [2] introduced a simple example from [53] which explains this problem. Consider a simple second-order system whose dynamics matrix can switch between two matrices

$$\dot{x} \in \{A_1 x,\ A_2 x\} \tag{2.8}$$

This can be viewed as an LPV plant with $\theta \in \langle -1, 1\rangle$. In [53] the authors shows that for

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -a \\ 1/a & -1 \end{pmatrix}$$

where $3+\sqrt{8} < a < 10$, the above system is stable for arbitrary switching, but no P exists satisfying conditions (2.7). This is called by some authors [54–57] as the conservativeness

[1] NP-hard (Non-deterministic Polynomial-time hard), in computational complexity theory, is a class of problems that are, informally, "at least as hard as the hardest problems in NP"

of the quadratic stability. Therefore a lot of people were looking for a solution on how to reduce the conservatism. And this has resulted in certain special structures of suitable Lyapunov functions [54–60].

Let denote
$$A(t,\tau;\theta([\tau,t])) \tag{2.9}$$
as the state transition matrix for an LPV system, where the dependence on the parameter trajectory is explicit (over the interval $[\tau, t]$). Accordingly

$$x(t) = A(t,\tau;\theta([\tau,t]))x(\tau) \tag{2.10}$$

Assuming that an LPV plant is exponentially stable for all parameter trajectories, there exist m and $\lambda > 0$ such that

$$\|A(t,\tau;\theta([\tau,t]))\| \leq me^{-\lambda(t-\tau)} \tag{2.11}$$

Let T be such that $me^{-\lambda T} < 1$, and define the following Lyapunov function candidate (e.g., [61])
$$V(x,t) = \int_t^{t+T} |A(\tau,t;\theta([t,\tau]))x|^2 d\tau \tag{2.12}$$
Fig. 2.3 illustrates the construction of this function. This function is the energy of the solution over the interval $[t, t+T]$. One can show that $V(x(t),t)$ is decreasing along solutions of the LPV system. In particular

$$V(x(t+h),t+h) - V(x(t),t) \approx -h|x(t)|^2 + h|x(t+T)|^2 < -\left(1 - me^{-\lambda T}\right)|x|^2 \tag{2.13}$$

Neglecting issues of differentiability, the above construction suggests that

$$\frac{dV(x(t),t)}{dt} < -c|x|^2 \tag{2.14}$$

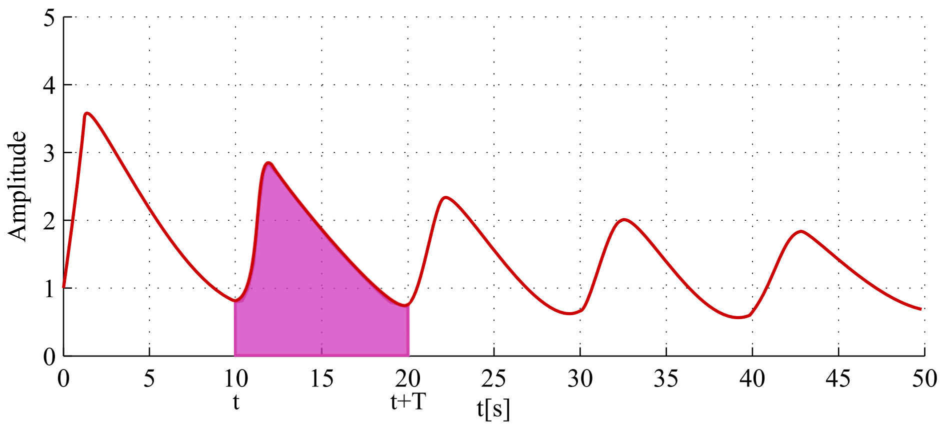

FIGURE 2.3: Integral for Lyapunov function construction

The structure of this Lyapunov function can be rewritten as a quadratic function in x, where the defining matrix is a function of the future parameter trajectory

$$V(x(t),t) = x^T(t)P(\theta([t,t+T]))x(t) \tag{2.15}$$

We can reparametrize the function to be a function of past parameter trajectories

$$V(x(t),t) = x^T(t)\tilde{P}(\theta([t-T,t]))x(t) \tag{2.16}$$

The authors in [59] used a similar construction to derive the following theorem

Theorem 2.3. *[59] An LPV system is exponentially stable for arbitrary time variations if and only if there exists a trajectory dependent quadratic Lyapunov function of the form*

$$V(x,t) = x^T P(\theta([t-T,t]))x \tag{2.17}$$

In discrete time, authors [59] use this result to derive a numerical search for Lyapunov functions. Regarding the previous discussion on complexity, this search may need to admit progressively longer intervals of trajectory dependence.

It turns out that one can eliminate the dependence on the parameter trajectory altogether. The intuition is as follows. From the Lyapunov function in *Theorem 2.3*, define

$$\bar{V}(x) = \inf_{\theta([t-T,t])} x^T P(\theta([t-T,t]))x \tag{2.18}$$

The new Lyapunov function is the former Lyapunov function evaluated at a *worst case trajectory* [2]. Again, an informal analysis illustrates that this parameter-independent Lyapunov function decreases along the solution of the LPV system for all parameter trajectories. This motivates the existence in general of a pseudo-quadratic Lyapunov function.

Authors Molchanov and Pyatnitskiy in [60] introduced the following theorem

Theorem 2.4. *[60] An LPV system is exponentially stable for arbitrary time variations if and only if there exists a Lyapunov function of the form*

$$V(x) = x^T P(x)x \tag{2.19}$$

for some family of matrices $P(\cdot)$, with the property that $P(\alpha x) = \alpha P(x)$ for $\alpha \geq 0$.

In book [2], papers [62, 63], survey [64] and monograph [65] we can find further discussions which go on to characterize alternative piecewise linear structures for exponentially stable LPV systems. Besides these we can find papers which investigate the stability of an LPV systems using parameter-dependent quadratic stability [54, 56, 57, 66, 67]. The main principle of parameter-dependent quadratic stability is that against the result with quadratic stability we have one Lyapunov function for all vertices of θ. So the Lyapunov function is parameter-dependent

$$V(\theta(t)) = x^T(t)P(\theta(t))x(t) \tag{2.20}$$

where $\theta \in \Omega$ and

$$P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i \qquad (2.21)$$

P. Gahinet, P. Apkarian and M. Chilali in [57] in this context introduced the following theorem

Theorem 2.5. *[57] The LPV system (2.1) for $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ is affinely quadratically stable if and only if there exist $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that*

$$P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i > 0 \qquad (2.22)$$

and for the first derivative of Lyapunov function $V(\theta) = x^T P(\theta)x$ along the trajectory of LPV system (2.1) it holds

$$\frac{dV(x,\theta)}{dt} = x^T \left(A(\theta)^T P(\theta) + P(\theta)A(\theta) + \frac{dP(\theta)}{dt} \right) x < 0 \qquad (2.23)$$

where

$$\frac{dP(\theta)}{dt} = \sum_{i=1}^{p} P_i\dot{\theta}_i \leq \sum_{i=1}^{p} P_i\rho_i$$

In this case we must have predefined the maximum rate of change of scheduled parameters $\dot{\theta}_i$ as ρ_i.

2.1.3 Summary

In this section (Linear parameter-varying system) the LPV systems were presented and described and their stability analysis since its introduction (1988 by Jeff. S. Shamma [1]) to the present (2015). The analysis and theorems stated herein are presented in an informal manner. Technical details may (and should) be found in the associated references.

2.2 Gain scheduling

The robust control theory is well established for linear systems but almost all real processes are more or less nonlinear. If the plant operating region is small, one can use the robust control approaches to design a linear robust controller where the nonlinearities are treated as model uncertainties. However, for real nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable. For this reason the controller design for nonlinear systems is nowadays a very determinative and important field of research.

Gain scheduling is one of the most common used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. In this section the main principles, several classical approaches and finally the linear parameter-varying based version of gain scheduling are presented and investigated.

2.2.1 Introduction to gain scheduling

In literature a lot of term are meant under gain scheduling (GS). For example switching or blending of gain values of controllers or models, switching or blending of complete controllers or models or adapt (schedule) controller parameters or model parameters according to different operating conditions. A common feature is the sense of decomposing nonlinear design problems into linear or nonlinear sub-problems. The main difference lies in the realization.

Consequently gain scheduling may be classified in different way

- According to decomposition
  1. GS methods decomposing nonlinear design problems into linear sub-problems
  2. GS methods decomposing nonlinear design problems into nonlinear (affine) sub-problems
- According to signal processing
  1. Continuous gain scheduling methods
  2. Discrete gain scheduling methods
  3. hybrid or switched gain scheduling methods
- According to main approaches
  1. Classical (linearization based) gain scheduling
  2. LFT based GS synthesis
  3. LPV based GS synthesis
  4. Fuzzy GS techniques
  5. Other modern GS techniques

2.2.1.1 History of gain scheduling

The ferret in the history of gain scheduling appears in the 1960s but a similar simpler technique was used in World War II toat control the rockets V2 (switching controllers based on measured data). It is not surprising therefore that gain scheduling as a concept or notion firstly appear in flight control and later in aerospace. Leith and Leithead in their survey [9] and likewise also Rugh and Shamma in their survey paper [10] considered the first appearance of GS from the 1960s. Rugh stated in his survey that " *Gain control*" does appear in the 25th Anniversary Index (1956–1981) published in 1981 but only one of the five listed papers is relevant to gain scheduling. Also *Automatica* lists gain scheduling as a subject in its 1963–1995 cumulative index published in 1995. Of the four citations given, only one dated earlier than 1990 [1]. It can be stated that increased attention to gain scheduling appeared after introducing the LPV paradigm by Jeff. S. Shamma (1988). This is partly understandable because LPV paradigm allowed to describe nonlinear system as a family of linear systems and hence investigate the stability of these systems. *Fig. 2.4* shows the major dates with remarks in a time-line of gain scheduling.

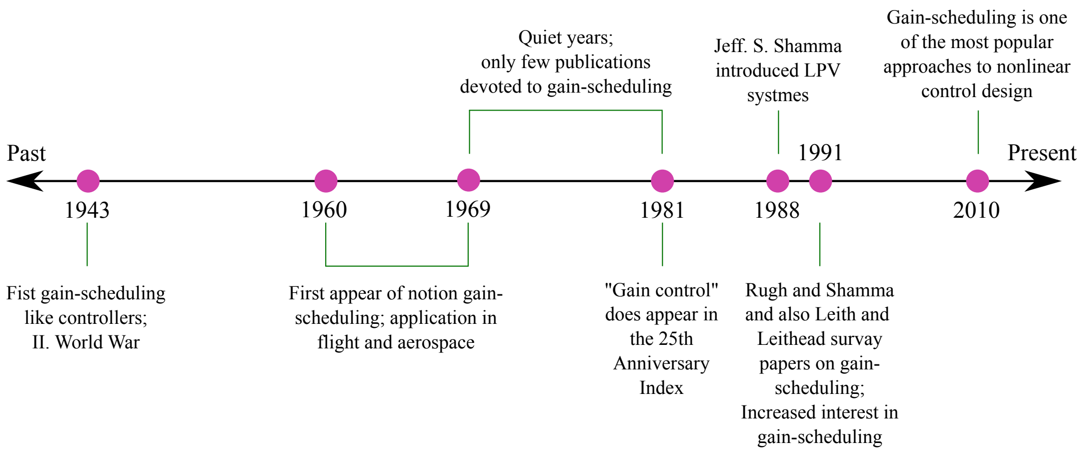

FIGURE 2.4: The time-line of gain scheduling

2.2.1.2 Application of gain scheduling

As already noted, traditionally the gain scheduling was the primary design approach to flight control and, consequently, many of the first articles and papers were associated with flight control [68–75] and aerospace [76–78]. Then gradually GS has been used almost everywhere in control engineering which was greatly advanced with the introduction of LPV systems.

The second big bang in the history of gain scheduling was the advent of fuzzy gain scheduling. Today, every second paper that appears under gain scheduling is devoted to fuzzy gain scheduling. Due to this wide range of gain-schedule approaches, gain scheduling is now used in several fields in practice. For example in power systems the gain scheduling enjoyed exceptional success in control of wind turbines [79–85]. But beside all this, some papers are devoted to hydro turbines [86, 87], gas turbines [88], power system stabilizers [89] and generators [90]. Many papers in gain scheduling are devoted to magnetic bearings [91–96] but we can find some papers devoted to also to microgravity [97], turbofan engine [98] and diabetes control [99].

2.2.2 Classical gain scheduling

In the case of nonlinear dynamics an idea is widely used among control engineers to linearize the plant around several operating points and to use linear control tools to design a controller for each of these points. The actual controller is implemented using the gain scheduling approach. Success of such an approach depends on establishing the relationship between a nonlinear system and a family of linear ones. There are two main problems:

1. Stability results: stability of the closed-loop nonlinear system and of the closed-loop family of linear systems, when scheduled parameters are changes.
2. Approximation results which provide a direct relationship between the solution of closed-loop nonlinear systems and the solution of associated linear systems [10], [9]

Rugh and Shamma in [10] comprise four main steps in classical gain scheduling

1. A family of LTI approximations are obtained from nonlinear plant at constant operating points (equilibria), parametrized by exogenous signal θ (scheduled parameter) which is computed using linearization based scheduling. The linearization has to correspond to zero error. Other syntheses to derive a parameter-dependent model are
   - *Off-equilibrium or velocity based linearization* [9, 100–102] - when zero equilibrium points or working conditions are not present
   - *Quasi LPV approach* [9, 10, 102], in which the plant dynamics are rewritten to distinguish nonlinearities as time-varying parameters that are used as scheduling variables.
   - *Direct LPV modelling, based on a linear plant incorporating time-varying parameters* [1, 75, 102] - when no nonlinear plant is involved. This also includes black-box or data-based modelling methods
2. A set of LTI controllers are designed using linear control tools for previously derived set of local LTI models to achieve specified performance and stability at each operating point. The resulting set of controllers is also parametrized by scheduled parameter θ. Although the scheduled parameter is time-varying, the classical gain scheduling methods are based on fixed or frozen scheduling parameter values. To enable subsequent scheduling, interpolation of controller parameters, the set of LTI controllers almost requires a fixed structure of the controller design. Exceptions are
   - in the case direct derivation of a Linear Parameter-Varying (LPV) controller for a corresponding LPV plant model is possible, subsequent scheduling, interpolation becomes superfluous.

- when discrete or hybrid scheduling instead of continuous scheduling is demanded, the set of controller designs not necessarily need to be fixed-structured.

3. Implementation of the family of LTI controllers such that the controller coefficients are scheduled according to the current value of the scheduling variable, e.g. by controller gain interpolation or scheduling. At this point, $\theta = \theta(t)$ is implemented. At each operating point, the scheduled controller has to be linearized to the corresponding linear controller design as well as provide a constant control value yielding zero error at these points. As mentioned in Step 2, in the case of direct scheduling, this step becomes superfluous. Furthermore, in the case of discrete scheduling, the implementation of the LTI controllers involves the design of a scheduled selection procedure that is applied to the set of LTI controllers, rather than the design of a family of scheduled controllers. The presence of hidden coupling terms is an important aspect which yields various additional requirements to the scheduling procedure.

4. Typically, local performance assessment can be performed analytically, whereas assessment of global performance and robustness has to be established by extensive simulations. Non-local performance of the gain-scheduled controller is evaluated and checked by simulations.

2.2.3 LFT and LPV based gain scheduling

The LPV and LFT syntheses are based on LPV and LFT plant representations respectively (Naus [102]). Both methods yield direct synthesis of a controller utilizing ($\mathcal{L}_2$ or H_∞) norm based methods, with guarantees the robustness, performance and nominal stability of the overall gain-scheduled design [7, 57, 66, 102–104]. LPV and LFT syntheses essentially involve only two main steps.

1. The first step corresponds to the classical approach. A family of LTI approximations of a nonlinear plant at constant operating points (equilibria), parameterized by constant values of convenient plant variables or exogenous signals θ are computed. Subsequent implementation of the controller requires $\theta = \theta(t)$ to be a measurable variable. Besides the already mentioned methods, which all arrive at Linear Parameter-Varying (LPV) models, in specific cases a LFT description is possible. The LFT description serves as a basis for subsequent LFT controller synthesis.

2. LPV and LFT control synthesis directly yield a gain-scheduled controller. Stability and performance specifications can be guaranteed a priori as the time-varying parameter $\theta(t)$ instead of its corresponding frozen value θ is addressed in the design process. In [102] one can find only continuous-time gain scheduling but the author Sato in 2011 [105] introduced discrete-time version of LPV based gain scheduling where stability investigated with both H_2 and H_∞.

2.2.3.1 LPV gain scheduling

The main advantages of the LPV control synthesis are as follows (Naus [102])

- There exists a solid theoretical foundation guaranteeing a priori stability and performance for all $\theta(t)$ given a corresponding range and rate of variation of $\theta(t)$
- The corresponding controller design is global with respect to the parametrized operating envelope Ω, Ω_t, whereas classical gain scheduling techniques focus on local system properties.
- A controller is synthesized directly, rather than its construction from a family of local linear controllers.

The main disadvantages are

- with respect to classical gain scheduling techniques, the controller synthesis is much more involved, which results in focusing on appropriate problem formulation rather than the actual controller design
- generally, conservatism has to be introduced to arrive at a feasible and convex problem
- with respect to classical gain scheduling, allowing for arbitrary linear controller design techniques, a predefined controller design synthesis has to be adopted.

As the latter point already indicates, LPV syntheses constitutes a specific performance evaluation framework, whereas classical gain scheduling provides an open framework. Typically, LPV syntheses employ the induced $\mathcal{L}_2$-norm as a performance measure, which is directly related to linear H_∞ techniques. As a result, an LPV control synthesis applied to a time-invariant system is equivalent to a standard H_∞ approach. In general, LPV control syntheses can be categorized into

1. Lyapunov-based approach (LPV-based techniques)
2. techniques exploiting the specific structure of systems with LFT parameter dependence, utilizing a small-gain approach, which are also determined as LFT approaches
3. a combination of the two preceding points, which Naus referred as 'mixed' LPV-LFT approaches

2.2.3.2 LFT gain scheduling

Exploiting LFT parameter-dependency of LPV systems enables application of a generalized H_∞ control synthesis using the optimally scaled small-gain theorem. The parameter variations are temporarily regarded as unknown perturbations [106]. Multipliers or scalings describing the nature of the unknown, time-varying parameter $\theta(t)$ are introduced to decrease conservatism with respect to LPV methods using a constant, common quadratic Lyapunov function only.

An LFT model is a special case of a LPV model, which is transformed using an upper LFT to $M - \Delta$ structure, where M is a constant and known part, and $\Delta(\theta(t))$ is the time-varying and unknown part. To enable an LFT representation, the system should have a rational dependence on the parameter θ and no poles in zero. Transformation of the LPV controller is similar using lower LFT, where K is the constant and known part and $\Delta_K(\theta(t))$ is the unknown part, where $\Delta_K(\theta(t))$ represents a possibly nonlinear controller scheduling function (see *Fig. 2.5*). The closed-loop interconnection of the resulting LFTs is transformed, again using a lower LFT

$$T = F_L\left(F_U(M, \Delta(\theta(t)))\; F_L(K, \Delta_K(\theta(t)))\right) \tag{2.24}$$

The closed-loop system matrix is defined as follows

$$\begin{bmatrix} \dot{\zeta}(t) \\ z_u(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_u & \mathcal{B}_p \\ \mathcal{C}_u & \mathcal{D}_u & \mathcal{D}_{up} \\ \mathcal{C}_p & \mathcal{D}_p u & \mathcal{D}_p \end{bmatrix} \begin{bmatrix} \zeta(t) \\ w_u(t) \\ w(t) \end{bmatrix} \tag{2.25}$$

where $w(t) \rightarrow z(t)$ is the performance channel and $z_u(t) = \Delta w_u(t)$. The closed-loop system has repeated linear fractional parameter dependence.

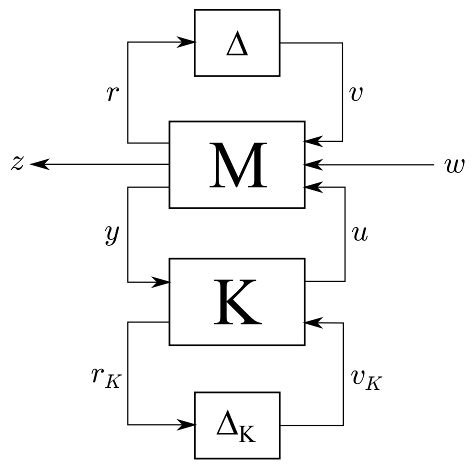

FIGURE 2.5: LFT $M - \Delta$ structure

Analogous to the previously presented LPV controller synthesis, the scaled small-gain control synthesis is based on the robust performance analysis of the closed-loop system. Consider the closed-loop LFT interconnection system which is represented by (2.24), (2.25). The corresponding analysis involves three main conditions regarding the closed-loop model T, comprising [107]

- well-posedness, i.e. for all initial conditions corresponding to the augmented system M and all inputs $w(t)$ to the system, the system should have a unique solution.
- exponential stability of the closed-loop system, which is analyzed via the small-gain theory, which states that a closed-loop system is stable provided that the loop-gain is less than unity
- performance; (quadratic performance or $\mathcal{L}_2$-performance or ...)

Combining all three conditions we obtain a set of LMIs, which is directly derived from the general LPV synthesis constraints [108].

The main advantages of the LFT approach are the possibility to use the optimally scaled small-gain theorem, which is well-established in literature, and the possibility to arrive at a finite dimensional set of LMIs using multipliers. However, the general full-block multiplier approach is complex, whereas relaxations on the multipliers introduce (much) conservatism. Furthermore, the use of a constant, common quadratic Lyapunov function, implying that infinitely fast parameter variations are accounted for, introduces conservatism, especially in the case of large parameter variations. Finally, the controller inherits the LFT parameter dependency, which might be conservative, and possible realness of the parameter $\theta(t)$ is not exploited, introducing conservatism as well.

2.2.4 Fuzzy gain scheduling

Fuzzy gain scheduling should overcome the disadvantage of classical gain scheduling regarding the restriction of stability and performance analysis to local rather than global closed-loop behaviour [70, 88, 96, 102]. The corresponding fuzzy modelling considers the transient dynamics of the original nonlinear model instead of local linearizations only. Fuzzy gain scheduling techniques may involve classical gain scheduling alike as well as LPV techniques. According to Naus [102], focusing on the former one, four main steps have to be considered.

1. Analogous to classical gain scheduling, sets of local LTI models and corresponding LTI controllers have to be designed. Focus lies on the regions of the envelope of operating conditions for which these controllers assure stability and desired performance of the resulting (local) closed-loop systems.
2. To arrive at a fuzzy model, so-called weighting or scaling functions are designed, corresponding to the before mentioned regions. Utilizing these weighting functions, the local models are blended. Specifying a specific approximation accuracy of the fuzzy model with respect to the original nonlinear model, yields the required number of local models.

3. The set of local controllers is blended analogous to the set of local models. The same weighting functions are utilized. The blending yields scheduling of the controller outputs rather than scheduling of the controllers or controller coefficients. Hence, members of the corresponding parameterized set of LTI controllers do not necessarily need to have fixed structure and dimension.

4. Stability and performance are established by extensive simulations, analogous to classical gain scheduling. However, in the case of fuzzy controller design, global as well as local specifications have to be derived from simulations, as the characteristic dynamics of the fuzzy model can not be related to the dynamics of the set of local models.

2.2.5 Summary

This section deals with gain scheduling and gain-scheduled controller designs (classical gain scheduling, LPV and LFT based gain scheduling and fuzzy gain scheduling).

The main advantage of classical gain scheduling is that it inherits the benefits of linear controller design methods, including intuitive classical design tools and time as well as frequency domain performance specifications. PID control is the most used control strategy in industrial applications due to its relatively simple and intuitive design, hence this is a major advantage with respect to other nonlinear controller design syntheses. The approach thus enables the design of low computational effort controllers. Conceptually, gain scheduling involves an intuitive simplification of the problem into parallel decompositions of the total system.

LPV and LFT synthesis require a true LPV model as a basis. In general however, gain scheduling may be employed in the absence of an analytical model, e.g. on the basis of a collection of plant linearizations. Consequently, controller design based on a whitebox as well as a blackbox and even data-based 'modeling' is possible. If the possibility of fast parameter variations is not addressed in the design process, guaranteed properties of the overall gain-scheduled design cannot be established. The main advantage of LPV and LFT control synthesis is that they do account for parameter variations in the controller design, which results in a priori guarantees regarding stability and performance specifications. The main drawback of LPV and LFT control synthesis involves conservativeness, which has to be introduced to enable solving the resulting LMIs. As a result of that, current LPV and LFT syntheses comprise specific extensions of robust control techniques rather than true generalizations. However, current and future research still provides and will provide less conservative solutions.

The main drawback of fuzzy gain scheduling involves the lack of a relation between the dynamic characteristics of the original nonlinear model and the fuzzy model. Even locally, the dynamics of the fuzzy model can not be related to the original nonlinear model. Fuzzy gain scheduling techniques may involve classical gain scheduling alike as well as LPV techniques.

The analysis and theorems stated herein are presented in an informal manner. Technical details may (and should) be found in the associated references.

2.3 Discussion

This thesis is devoted to gain scheduling within this to LPV based gain scheduling because in our opinion the biggest potential between gain scheduling approaches is in the LPV based gain scheduling. Despite this, we described all main historical approaches to gain scheduling as classical gain scheduling, LFT based gain scheduling and novel fuzzy gain scheduling.

As we mentioned LPV based gain scheduling appear in 1988 when Jeff. S. Shamma introduced the LPV paradigm in his Ph.D. thesis [1]. Today LPV paradigm has become a standard formalism in systems and controls with lots of researches and articles devoted to analysis, controller design and system identification of these models. Due to this nowadays the LPV gain scheduling belongs to the most popular approaches to nonlinear control design. But, as we mentioned in *Chapter 1*, there are still a lot of unsolved problems. Browsing through literature we cannot find any general LPV based gain-scheduled approach which will involve guaranteed cost and affine quadratic stability. In addition there are very rudimentary approaches in switched and predictive control not to mention the robust and discrete design approaches.

Currently, nowhere it is solved how to affect the performance quality separately in each working point when direct LPV controller approach is used. Furthermore, there are only few papers devoted to output feedbacks and they also not use fixed order output feedbacks like PID/PSD controllers.

Most of the papers devoted to LPV based gain scheduling convert the stability conditions into LMI problems. But currently we cannot find a general LMI approach with guaranteed stability and guaranteed cost. Furthermore, nowhere it is solved how to consider input/output constraints without need of on-line optimization.

Among other things, to find a solution for some of these unsolved problems (deficiencies) were the main goals of the research which is summarized in this thesis.

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3

Summary of included papers

In this chapter one can find an overview of research results with a brief presentation of included papers, which covers the main research results which was obtained during the last 2.5 years. The papers have been reformatted for uniformity but otherwise they are unchanged.

3.1 Introduction

In the initial stage of our research I with my supervisor Prof. Ing. Vojtech Veselý, DrSc. developed a gain-scheduled controller design which guarantees the closed-loop system stability and a guaranteed cost for continuous-time linear parameter-varying (LPV) systems for all scheduled parameter changes with pre-defined rate of scheduled parameter changes. These results were published in Journal of Process Control and presented at several conferences (ICCC'13, ICPC'13, ELITECH'13, IN-TECH'13). After that we expanded this theory to robust controller design for continuous and discrete-time uncertain LPV systems with possibility for variable weighting in cost function. Some of these results were published in Journal of Electrical Engineering, in Journal of Electrical Systems and Information Technology and they were presented at several conferences such as the European Control Conference 2014 (ICCC'14, CPS'14, SSKI'14, ELITECH'14, ICPC'15).

In the middle stage of our research we modified our approaches from BMI (bilinear matrix inequality) to LMI (linear matrix inequality) problem. This caused that our controller synthesis works for high-order systems (50-60th order was tested). We successfully ported our approaches to switched and model predictive controller design. Some of these results have been published in Journal of the Franklin Institute, in Journal of Electrical Engineering, in International Review of Automatic Control, in Asian Journal of Control and will be presented at several IFAC symposiums and conferences as MICNON'15 or ROCOND'15 (ICPC'15, ELITECH'15). Also some of these results are under review process in journals International Journal of Control, Automation and Systems and in Archives of Control Sciences.

In the final stage we successfully developed a new stability condition where we could bypass the multi-convexity that significantly reduced the conservativeness of the controller synthesis. In addition we added to the controller synthesis the input (hard) / output (soft) constraints as well as input rate (soft) / output rate (soft) constraints where we do not need online optimization. Publications of these results are only in the preliminary stage but hopefully they will be published in high impact factor journals, too.

In the following sections one can find a brief presentation of the included papers.

3.2 Summary of included papers

Paper 1

V. Veselý, A. Ilka, Gain-scheduled PID controller design, Journal of Process Control, 23 (8) (2013) 1141–1148.

In the first paper one can find a linear parameter-varying based gain-scheduled controller design which guarantees the closed-loop stability and the guaranteed cost for all scheduled parameter changes. The proposed procedure is based on the Lyapunov theory of stability, guaranteed cost and on the concept of multi-convexity. In the gain-scheduled controller design procedure one can include the maximal value of the rate of gain-scheduled parameter changes, which allows to decrease conservativeness and obtain the controller with a given performance. The main results for the case of gain-scheduled closed-loop stability analysis reduce to LMI (linear matrix inequality) condition and for gain-scheduled controller synthesis to BMI (bilinear matrix inequality) one. One can use a free and open source BMI solver PenLab. Another advantage of this method is the fact that we can affect the quality and cost with weighting matrices R, Q, S. Numerical examples illustrate the effectiveness of the proposed approach.

Paper 2

A. Ilka, V. Veselý, Gain-Scheduled Controller Design: Variable Weighting Approach, Journal of Electrical Engineering, 65 (2) (2014) 116-120.

In the second paper one can find a linear parameter-varying based gain-scheduled controller design which guarantees the closed-loop stability and the parameter-varying guaranteed cost for all scheduled parameter changes. The proposed procedure is also based on the Lyapunov theory of stability, guaranteed cost and on the concept of multi-convexity. In the gain-scheduled controller design procedure one can include also the maximal value of the rate of gain-scheduled parameter changes, which allows to decrease conservativeness. To access the performance quality a new quadratic cost function is used, where weighting matrices are time varying and depends on scheduled parameter. Using these original variable weighting matrices we can affect performance quality separately in each working points and we can tune the system to the desired condition through all parameter changes. The class of control structure includes decentralised fixed order output feedbacks like PID controller. Numerical examples illustrate the effectiveness of the proposed approach.

Paper 3

V. Veselý, A. Ilka, Design of robust gain-scheduled PI controllers, Journal of the Franklin Institute, 352 (2015) 1476-1494.

In this paper a novel methodology for robust gain-scheduled controller design is presented. The proposed design procedure is based on the robust stability condition developed for an uncertain LPV system model introduced in this paper. The obtained results, illustrated on examples, show the applicability of the designed robust gain-scheduled controller and its ability to cope with polytopic model uncertainties. Several forms of parameter dependent/quadratic Lyapunov functions are presented and tested by simulations. Though the proposed robust controller design approach with parameter dependent Lyapunov function does not consider quick changes of parameters (either uncertainty or gain scheduling), simulation results prove the potential ability of the designed closed-loop to withstand also these changes. The obtained results are in the form of BMI and LMI approaches. The proposed approach contributes to the design tools for robust gain-scheduled controllers. The obtained design results and their properties are illustrated on simulation examples.

Paper 4

V. Veselý, A. Ilka, Robust Gain-Scheduled PID Controller Design for Uncertain LPV Systems, Journal of Electrical Engineering, 66 (1) (2015) 19-25.

A novel methodology is proposed for robust gain-scheduled PID controller design for uncertain LPV systems. The proposed design procedure is based on the parameter-dependent quadratic stability approach. A new uncertain LPV system model has been introduced in this paper. To access the performance quality, the approach of a parameter varying guaranteed cost is used which allowed to reach the desired performance for different working points. Several forms of parameter dependent quadratic stability are presented which withstand arbitrarily fast model parameter variation or/and arbitrarily fast gain-scheduled parameter variation. Numerical examples show the benefit of the proposed method.

Paper 5

A. Ilka, I. Ottinger, T. Ludwig, M. Tárník, V. Veselý, E. Miklovičová, J. Murgaš, Robust Controller Design for T1DM Individualized Model: Gain Scheduling Approach, International Review of Automatic Control (I.RE.A.CO.), (2) (2015)

This paper deals with the robust gain-scheduled controller design for individualized type 1 diabetes mellitus (T1DM) subject model. The controller is designed using LPV model created from T1DM minimal model with two additional subsystems - absorption of digested carbohydrates and subcutaneous insulin absorption. Data collected from continuous glucose monitoring with the help of pharmacodynamics and pharmacokinetics characteristics were used for model identification. The closed-loop stability and cost for all scheduled parameters is guaranteed by the controller design approach. In contrast to publications in literature we presented a completely new LPV description of Bergman's minimal model and a new approach to controller design. The obtained design procedure can be used in systems where we need to save the operation energy (e.g. low-cost micro-controllers). The benefits of the presented approach are shown in the simulation results.

Paper 6

V. Veselý, A. Ilka, Novel approach to switched controller design for linear continuous-time systems, Accepted in Asian Journal of Control, Acceptance day: 22. May, 2015.

In this paper one can find a novel approach to the design of an output feedback switched controller with an arbitrary switching algorithm for continuous-time invariant systems which is described by a novel plant model as a gain-scheduled plant using the multiple quadratic stability and quadratic stability approaches. In the proposed design procedure there is no need to use the notion of the "dwell-time". The obtained results are in the form of bilinear matrix inequalities (BMI). Numerical examples show that in the proposed method the design procedure is less conservative and gives more possibilities than that described in the papers published previously.

Paper 7

V. Veselý, A. Ilka, Robust Switched Controller Design for Nonlinear Continuous Systems, Accepted for publication and for presentation at 1st IFAC conference on Modelling and Control of Nonlinear Systems (MICNON'15), Saint Petersburg, Russian Federation, June 24-26, 2015.

A novel approach is presented to robust switched controller design for nonlinear continuous-time systems under an arbitrary switching signal using the gain scheduling approach. The proposed design procedure is based on the robust multi parameter dependent quadratic stability condition. The obtained switched controller design procedure for nonlinear continuous-time systems is in the bilinear matrix inequality form (BMI). The obtained results, illustrated on examples, show the applicability of the designed switched robust gain-scheduled controller and its ability to cope with model uncertainties. In the paper several forms of parameter dependent/quadratic Lyapunov functions are proposed. The properties of the obtained design are illustrated on simulation examples.

Paper 8

A. Ilka, V. Veselý, Gain-Scheduled MPC Design for Nonlinear Systems with Input Constraints, Accepted for publication and for presentation at 1st IFAC conference on Modelling and Control of Nonlinear Systems (MICNON'15), Saint Petersburg, Russian Federation, June 24-26, 2015.

A novel methodology is proposed for discrete model predictive gain-scheduled controller design for nonlinear systems with input(hard)/output(soft) constraints for finite and infinite prediction horizons. The proposed design procedure is based on the linear parameter-varying (LPV) paradigm, affine parameter-dependent quadratic stability and on the notion of the parameter-varying guaranteed cost. The design procedure is in the form of BMI (we can use a free and open source BMI solver). Numerical examples show the benefits for the finite and infinite prediction horizon.

Paper 9

V. Veselý, A. Ilka, Unified Robust Gain-Scheduled and Switched Controller Linear Continuous-Time Systems, Submitted to International Review of Automatic Control (I.RE.A.CO), Submitted on 25. May, 2015.

The proposed paper addresses the problem how to obtain the new unified procedure to design a robust gain-scheduled and switched controller with arbitrary switching for continuous-time systems described by a novel robust plant model using the parameter dependent quadratic stability (PDQS) approach. The obtained unified controller design procedure ensures the closed-loop stability and guaranteed cost for a prescribed rate of change of the system switching (gain-scheduled) variable. In some real cases the rate of change of the switching signal is finite. This assumption was used in the paper to obtain the switched controller design procedure. Numerical examples illustrate the effectiveness of the proposed approach.

4

Gain-scheduled PID controller design (Paper 1)

Abstract

Gain scheduling (GS) is one of the most popular approaches to nonlinear control design and it is known that GS controllers have a better performance than robust ones. Following the terminology of control engineering, linear parameter-varying (LPV) systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters. Our approach is based on considering that the LPV system, scheduling parameters and their derivatives with respect to time lie in a priori given hyper rectangles. To guarantee the performance we use the notion of guaranteed costs. The class of control structure includes centralized, decentralized fixed order output feedbacks like PID controller. Numerical examples illustrate the effectiveness of the proposed approach.

Keywords: Gain-scheduled control, controller design, structured controller, decentralized control, MIMO LPV systems.

4.1 Introduction

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. Linear parameter varying (LPV) systems have the following interpretations:

– they can be viewed as linear time invariant (LTI) plants subject to time-varying known parameters $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle$,

– they can be models of linear time-varying plants,

– they can be LTI plant models resulting from linearization of the nonlinear plants along trajectories of the parameter $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle$ which can be measured.

For the first and third class of systems, parameter θ can be exploited for the control strategy to increase the performance of closed-loop systems. Hence, in this paper the following LPV system will be used:

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u \\ y &= Cx \end{aligned} \tag{4.1}$$

where for the affine case

$$A(\theta(t)) = A_0 + A_1\theta_1(t) + \ldots + A_p\theta_p(t) \tag{4.2}$$

$$B(\theta(t)) = B_0 + B_1\theta_1(t) + \ldots + B_p\theta_p(t) \tag{4.3}$$

and $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control input, $y \in \mathbb{R}^l$ is the measurement output vector, A_0, B_0, A_i, B_i, $i = 1, 2 \ldots p$, C are constant matrices of appropriate dimension, $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle \in \Omega$ and $\dot{\theta}(t) \in \langle \underline{\dot{\theta}}\ \overline{\dot{\theta}} \rangle \in \Omega_t$ are vectors of time-varying plant parameters which belong to the known boundaries.

In the case of nonlinear dynamics a widely used idea among control engineers is to linearize the plant around several operating points and to use linear control tools to design a controller for each of these points. The actual controller is implemented using the gain scheduling approach. Success of such an approach depends on establishing the relationship between a nonlinear system and a family of linear ones. There are two main problems:

1. Stability results: stability of the closed-loop nonlinear system and of the closed-loop family of linear systems, when scheduled parameters are changes.
2. Approximation results which provide a direct relationship between the solution of closed-loop nonlinear systems and the solution of associated linear systems [1], [2]

The main motivation for our work lies in [3], [4], [5], [6], [7], [8], where in [3] the LPV controller is designed using the bounded real lemma for continuous and discrete time LPV systems such as to guarantee H_∞ performance.

Paper [4] discusses extensions of H_∞ synthesis techniques to allow for controller dependence on time-varying but measured parameters. In this case a higher performance can be achieved by control laws that incorporate measurements of θ to the control algorithm. Main results can be formulated as follows: Find a control structure such that the LPV controller satisfies closed-loop stability and minimizes of the induced L_2 norm of corresponding closed-loop systems. The author's approach [5] uses a bounding technique based on parameter-dependent Lyapunov function for design of PD controllers. Note that if LPV synthesis problem is solvable, then the induced L_2-norm of the closed-loop system is less than some given constant. The proposed approach represents generalization of the standard sub-optimal H_∞ control problem. In paper [6] the author shows that the performance of LPV systems with LPV controller can be improved by combining this

LMI method with MPC techniques and optimizing the H_2 (H_∞) norm. The author [8] tackles the design problem of gain scheduled controllers for LPV systems via parameter-dependent Lyapunov function. The author proposed a new design method as a set LMIs with single line search parameters. The author tackles two problems: H_∞ type problem and H_2. Recently, [9] proposed the design method for the gain scheduled problem using a similar technique to [8]. In the above paper the LPV controller is given in time domain with the same or lower order than the LPV systems using H_∞ optimization approach. The gain scheduling controller design for discrete-time systems is given in [10]. Paper [11] presents the design of gain-scheduled PI controller, when the uncertainty of the system is assumed to be the difference between the nonlinear model and the nominal linear model. PI controller is designed using quadratic Lyapunov H_∞ performance where index γ is H_∞ norm of closed-loop system, considered as closed-loop performance measure. Minimizing γ via LMI the gain scheduled controller is obtained. In [12] the authors design a novel gain scheduling controller for synchronous generator. Improved stability analysis and gain scheduled controller synthesis for parameter-dependent systems are proposed in [7]. Sufficient conditions for robust stability as well as conditions for the existence of a gain-scheduled controller are given in terms of a set of LMIs. The author's approach is based on the notion of quadratic stability and linear fractional representation for parameter dependent systems. The survey of scheduled controller analysis and synthesis can be found in excellent papers [1] and [2].

In this paper our approach is based on:

- A consideration of the LPV systems (4.1). The scheduling parameters θ_i, $i = 1, 2, \dots p$ and their derivatives with respect to time are supposed to lie in a priori given hyper rectangles.
- Affine quadratic stability (AQS) introduced by [13].
- To guarantee the performance we use the notion of guaranteed cost to optimize the given cost function.
- The class of control structure includes centralized, decentralized fixed order output feedback like PID controller.

The gain-scheduled controller design procedure is in the form of BMI. A feasible solution for closed-loop system ensures the affine quadratic stability [13] and guaranteed cost when the performance is defined in Q, R, S structure (see eq. (4.10)).

Quadratic stability (one Lyapunov function with one constant positive definite matrix cover all affine controller design procedure) is more conservative than AQS in general. AQS (Lyapunov function has an affine structure like (4.2)) incorporates information about the rate of variation $\dot{\theta}(t)$ to reduce conservatism. As we mentioned, in this paper the AQS approach will be used.

Our notations are standard. $D \in \mathbb{R}^{m\times n}$ denotes the set of real $m \times n$ matrices. I_m is an $m \times m$ identity matrix. If the size can be determined from the context, we will omit the subscript. $P > 0$ ($P \geq 0$) is a real symmetric, positive definite (semi-definite) matrix.

The paper is organized as follows. *Section 4.2* brings preliminaries and problem formulation. The main result is presented in *Section 4.3*. In *Section 4.4*, numerical examples illustrate the effectiveness of the proposed approach.

4.2 Preliminaries and problem formulation

Suppose that the state-space representation of an LPV system with p independent scheduling parameters is governed by (4.1). The scheduling parameters θ_i and their derivatives with respect to time $\dot{\theta}_i$ are supposed to lie in given hyper rectangles Ω and Ω_t, respectively. For design of the I part of the controller system, equation (4.1) has to be augmented, see [14] and example 1. Without change of notation the new augmented matrices dimensions are $A(\theta) \in \mathbb{R}^{(n+l)\times(n+l)}$, $B(\theta) \in \mathbb{R}^{(n+l)\times m}$, $C \in \mathbb{R}^{2l\times 2l}$ and $C_d \in \mathbb{R}^{l\times l}$ is the output matrix for D part of controller. The output feedback gain-scheduled control law is considered for PID controller in the form

$$u(t) = F(\theta)y + F_d(\theta)\dot{y}_d = F(\theta)Cx + F_dC_d\dot{x} \tag{4.4}$$

where $y_d = C_dx$ is the output feedback for the D part of the controller,

$$F(\theta) = F_0 + \sum_{i=1}^{p} F_i\theta_i \quad \in \mathbb{R}^{m\times 2l} \tag{4.5}$$

is the static output feedback gain scheduled matrix for the PI controller and

$$F_d(\theta) = F_{d_0} + \sum_{i=1}^{p} F_{d_i}\theta_i \quad \in \mathbb{R}^{m\times m} \tag{4.6}$$

is the static output feedback gain scheduled matrix for the D part of controller.

Remark 4.1. Since the reference signal does not influence the closed-loop stability, we assume that it is equal to zero.

Remark 4.2. If the derivative part of the controller includes some filter, the model of this filter can be included in the system model.

The closed-loop system is then

$$[I - B(\theta)F_d(\theta)C_d]\dot{x} = [A(\theta) + B(\theta)F(\theta)C]x \tag{4.7}$$

$$A_d(\theta)\dot{x} = A_c(\theta)x \tag{4.8}$$

$$\dot{x} = A_{cd}(\theta)x \tag{4.9}$$

where

$$A_{cd}(\theta) = A_d(\theta)^{-1}A_c(\theta)x$$
$$A_d(\theta) = I - B(\theta)F_d(\theta)C_d$$
$$A_c(\theta) = A(\theta) + B(\theta)F(\theta)C$$

It is well known that the fixed order dynamic output feedback control design problem is a special case of the static output feedback problem. To access the performance quality a quadratic cost function [15] known from LQ theory is often used in the form

$$J = \int_0^\infty (x^T Qx + u^T Ru + \dot{x}^T S\dot{x})dt \tag{4.10}$$

with $Q = Q^T \geq 0$, $R > 0$ and $S = S^T \geq 0$. The guaranteed cost is defined in a standard way.

Definition 4.1. Consider system (4.1) with control algorithm (4.4). If there exists a control law u^* and a positive scalar J^* such that the closed-loop system (4.7) is stable and the value of closed-loop cost function (4.10) satisfies $J \leq J^*$, then J^* is said to be a guaranteed cost and u^* is said to be guaranteed cost control law for system (4.1). □

Definition 4.2. [13] The linear closed-loop system (4.7) for $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ is affinely quadratically stable if and only if there exist $p + 1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that

$$P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i > 0 \tag{4.11}$$

and for the first derivative of Lyapunov function $V(\theta) = x^T P(\theta)x$ along the trajectory of closed-loop system (4.7) it holds

$$\frac{dV(x,\theta)}{dt} = x^T \left(A_{cd}(\theta)^T P(\theta) + P(\theta)A_{cd}(\theta) + \frac{dP(\theta)}{dt} \right) x < 0 \tag{4.12}$$

where

$$\frac{dP(\theta)}{dt} = \sum_{i=1}^{p} P_i\dot{\theta}_i \leq \sum_{i=1}^{p} P_i\rho_i$$

□

From LQ theory we introduce the well known results.

Lemma 4.1. *Consider the closed-loop system (4.7). Closed-loop system (4.7) is affinely quadratically stable with guaranteed cost if and only if the following inequality holds*

$$B_e = \min_u \left\{ \frac{dV(\theta)}{dt} + x^T Qx + u^T Ru + \dot{x}^T S\dot{x} \right\} \leq 0 \tag{4.13}$$

for all $\theta \in \Omega$ *and* $\dot{\theta} \in \Omega_t$ □

4.3 Main results

In this section the gain scheduled controller design procedure which guarantees the affine quadratic stability and guaranteed cost for $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ is presented. The main results for the case of gain scheduled closed-loop stability analysis reduce to LMI condition and for gain scheduled controller synthesis to BMI one.

The main result of this section, the gain scheduled design procedure, relies in the concept of multi-convexity, that is, convexity along each direction θ_i of the parameter space. The implications of multiconvexity for scalar quadratic functions are given in the next lemma [13].

Lemma 4.2. *Consider a scalar quadratic function of $\theta \in \mathbb{R}^p$.*

$$f(\theta_1, \ldots, \theta_p) = a_0 + \sum_{i=1}^{p} a_i\theta_i + \sum_{i,j=1}^{p} b_{ij}\theta_i\theta_j + \sum_{i=1}^{p} c_i\theta_i^2 \tag{4.14}$$

and assume that $f(\theta_1, \ldots, \theta_p)$ is multi-convex, that is

$$\frac{\partial^2 f(\theta)}{\partial \theta_i^2} = 2c_i \geq 0 \tag{4.15}$$

for $i = 1, 2, \ldots, p$. Then $f(\theta)$ is negative for all $\theta \in \Omega$ if and only if it takes negative values at the corners of θ. □

Using Lemma 4.2 the following theorem is obtained

Theorem 4.1. *Closed-loop system (4.7) is AQS with guaranteed cost if there exist $p+1$ definite matrices $P_0, P_1, P_2, \ldots, P_p$ such that $P(\theta)$ (4.11) is positive defined for all $\theta \in \Omega$, matrices N_1, N_2, Q, R, S and controller gain scheduled matrices $F(\theta)$ and $F_d(\theta)$, satisfying*

$$M(\theta) < 0; \quad \theta \in \Omega \tag{4.16a}$$

$$M_{ii} \geq 0; \quad i = 1, 2, \ldots, p \tag{4.16b}$$

where

$$M(\theta) = M_0 + \sum_{i=1}^{p} M_i\theta_i + \sum_{i=1}^{p}\sum_{j=1}^{p} M_{ij}\theta_i\theta_j$$

$$M_0 = \begin{bmatrix} W_{110} & W_{120} \\ W_{120}{}^T & W_{220} \end{bmatrix}$$

$$M_i = \begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}{}^T & W_{22i} \end{bmatrix}$$

$$M_{ij} = \begin{bmatrix} W_{11ij} & W_{12ij} \\ W_{12ij}{}^T & W_{22ij} \end{bmatrix}$$

$$
\begin{aligned}
W_{110} &= N_1 A_{d0} + A_{d0}^T N_1^T + C_d^T F_{d0}^T R F_{d0} C_d + S \\
W_{11i} &= N_1 A_{di} + A_{di}^T N_1^T + C_d^T F_{d0}^T R F_{di} C_d \\
&\quad + C_d^T F_{di}^T R F_{d0} C_d \\
W_{11ij} &= N_1 A_{dij} + A_{dij}^T N_1^T + C_d^T F_{di}^T R F_{dj} C_d \\
W_{120} &= P_0 + A_{d0}^T N_2^T - N_1 A_{c0} + C_d^T F_{di}^T R F_0 C \\
W_{12i} &= P_i + A_{di}^T N_2^T - N_1 A_{ci} + C_d^T F_{d0}^T R F_i C \\
&\quad + C_d^T F_{di}^T R F_0 C \\
W_{12ij} &= A_{dij}^T N_2^T - N_1 A_{cij} + C_d^T F_{di}^T R F_j C \\
W_{220} &= \sum_{k=1}^{p} P_k \rho_k - N_2 A_{c0} - A_{c0}^T N_2^T + Q \\
&\quad + C^T F_0^T R F_0 C; \rho_k \in \Omega_t \\
W_{22i} &= -N_2 A_{ci} - A_{ci}^T N_2^T + C^T F_0^T R F_i C \\
&\quad + C^T F_i^T R F_0 C \\
W_{22ij} &= -N_2 A_{cij} - A_{cij}^T N_2^T + C^T F_i^T R F_j C \\
A_{c0} &= A_0 + B_0 F_0 C \\
A_{ci} &= A_i + B_0 F_i C + B_i F_0 C \\
A_{cij} &= B_i F_j C \\
A_{d0} &= I - B_0 F_{d0} C_d \\
A_{di} &= -B_0 F_{di} C_d - B_i F_{d0} C_d \\
A_{dij} &= -B_i F_{dj} C_d
\end{aligned}
$$

□

Proof. Proof is based on *Lemmas 4.1* and *4.2*. From (4.8) and (4.12) we can obtain

$$[2N_1\dot{x} + 2N_2 x]^T [A_d(\theta)\dot{x} - A_c(\theta)x] = 0 \tag{4.17}$$

and

$$\frac{dV}{dt} = \dot{x}^T P(\theta) x + x^T P(\theta)\dot{x} + x^T \dot{P}(\theta) x \tag{4.18}$$

Summarizing the above two equations, for the time derivative of Lyapunov function one obtains

$$\frac{dV}{dt} = z^T \begin{bmatrix} N_1 A_d(\theta) + A_d(\theta)^T N_1^T & -N_1 A_c(\theta) + A_d^T N_2^T + P(\theta) \\ * & -N_2 A_c(\theta) - A_c^T(\theta) N_2^T + \sum_{i=1}^{p} P_i \rho_i \end{bmatrix} z \tag{4.19}$$

where $N_1, N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices and $z^T = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix}$. When one substitutes control algorithm (4.4) to the right hand side of (4.13) and then the obtained result is combined with (4.19) and substituted to (4.13), after some manipulation, using *Lemma 4.2* we obtain (4.16), which proofs the *Theorem 4.1*. □

Let us denote $\theta_m = \sum_{i=1}^{p} \theta_i$, multiplying (4.16a) with $\sum_{i=1}^{p} \frac{\theta_i}{\theta_m}$ assuming that $\theta_m \neq 0$ and $\theta_m \in \langle \underline{\theta}_m, \overline{\theta}_m \rangle$ we obtain

$$\frac{M_0}{\theta_m^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \theta_i \theta_j + \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{M_i}{\theta_m} \theta_i \theta_j + \sum_{i=1}^{p} \sum_{j=1}^{p} M_{ij} \theta_i \theta_j < 0 \tag{4.20}$$

After small manipulation

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \left[M_0 + M_i \theta_m + M_{ij} \theta_m^2 \right] \theta_i \theta_j < 0 \tag{4.21}$$

The closed-loop system will be stable or (4.21) holds if

$$K_{ij} + K_{ji} < 0, \quad i = 1, 2, \ldots, p, \quad j = i, i+1, \ldots, p \tag{4.22}$$

where

$$K_{ij} = M_0 + M_i \theta_m + M_{ij} \theta_m^2$$

Using stability conditions (4.22) and *Lemma 4.2* if the following inequalities are met, the closed-loop system is affine quadratically stable

$$\begin{aligned} 2M_0 + (M_i + M_j)\, \underline{\theta}_m + (M_{ij} + M_{ji})\, \underline{\theta}_m^2 &< 0 \\ 2M_0 + (M_i + M_j)\, \overline{\theta}_m + (M_{ij} + M_{ji})\, \overline{\theta}_m^2 &< 0 \\ M_{ij} + M_{ji} &\geq 0 \end{aligned} \tag{4.23}$$

for $i = 1, 2, \ldots, p$, $j = i, i+1, \ldots, p$.

Lemma 4.3. *Closed-loop system (4.7) is AQS with guaranteed cost if there exist $p+1$ definite matrices $P_0, P_1, \ldots, P_p$ such that for all $\theta \in \Omega$, $P(\theta)$ (4.11) is positive definite, matrices N_1, N_2 and gain scheduled matrices $F(\theta)$ and $F_d(\theta)$ are satisfying (4.23).* □

If the solution of *Theorem 4.1* or *Lemma 4.3* are feasible:

- For the case of closed-loop system stability, with respect to matrices N_1, N_2 and positive definite matrix $P(\theta)$ the closed-loop system is affine quadratically stable with guaranteed cost and for $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$. For this case gain matrices (4.4), (4.5) and (4.6) are known and (4.16), (4.23) reduces to LMI.

- For the gain-scheduled controller design with respect to matrices $F(\theta)$, $F_d(\theta)$, N_1, N_2 and positive definite matrix $P(\theta)$, the closed-loop system is affine quadratically stable with guaranteed cost and for $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$. For this case (4.16) and (4.23) are BMI.

4.4 Examples

The first example is taken from paper [16]. Consider a simple linear time-varying plant with parameter varying coefficients

$$\begin{aligned}\dot{x}(t) &= a(\alpha)x(t) + b(\alpha)u(t)\\ y(t) &= x(t)\end{aligned} \tag{4.24}$$

where $\alpha(t) \in \mathbb{R}$ is an exogenous signal that changes the parameters of the plant as follows

$$a(\alpha) = -6 - \frac{2}{\pi}\arctan\left(\frac{\alpha}{20}\right) \tag{4.25}$$

$$b(\alpha) = \frac{1}{2} + \frac{5}{\pi}\arctan\left(\frac{\alpha}{20}\right) \tag{4.26}$$

Let the problem be the design of a gain scheduled PID controller which will guarantee the closed-loop stability and guaranteed cost for $\alpha \in \langle 0, 100\rangle$. We will demonstrate that with the gain-scheduled controller we will obtain practically identical behaviour for closed-loop system. To be able to demonstrate this feature, let us divide the working area to 3 sections with 4 transfer functions in points $\alpha = 0, 15, 50, 100$ so that in each area, where the plant parameter changes, they are nearly linear (*Fig. 4.1*).

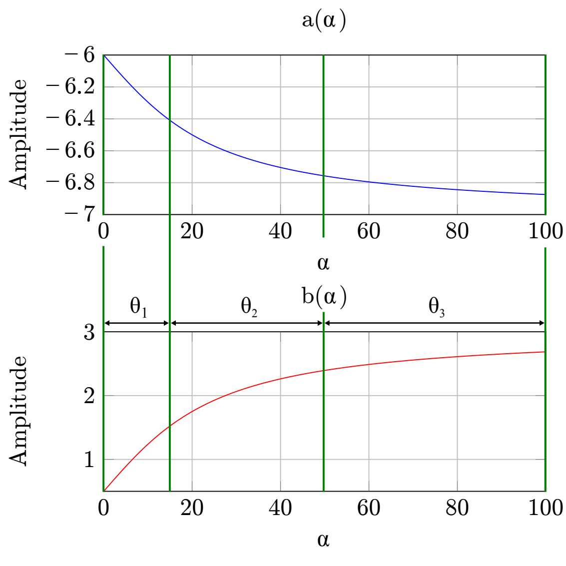

FIGURE 4.1: Exogenous signal $\alpha(t)$

In these working points the calculated transfer functions are:

$$\begin{aligned}G_{s1}|_{\alpha=0} &= \frac{0.5}{s+6}, & G_{s2}|_{\alpha=15} &= \frac{1.5242}{s+6.4097}\\ G_{s3}|_{\alpha=50} &= \frac{2.0642}{s+6.6257}, & G_{s4}|_{\alpha=100} &= \frac{2.6858}{s+6.8743}\end{aligned} \tag{4.27}$$

We transform the above transfer functions to the time domain to obtain the scheduling model in the form (4.1). The obtained model was extended for the gain-scheduled PID controller design. The extended model is given as follows

$$A_0 = \begin{bmatrix} -6.4370 & 0 \\ 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -0.2050 & 0 \\ 0 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -0.1080 & 0 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} -0.1240 & 0 \\ 0 & 0 \end{bmatrix}$$
$$B_0 = \begin{bmatrix} 1.5930 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5120 \\ 0 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0.2700 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.3110 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0$$

Using *Theorem 4.1* for $\theta_i \in \langle -1, 1 \rangle$, $i = 1, 2, 3$ we obtain gain scheduled controller in the form:

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 + G_{r3}\theta_3 \tag{4.28}$$

where

$$G_{r0} = \frac{0.4386s^2 + 2.8850s + 4.4678}{s}$$
$$G_{r1} = -\frac{1.72 \times 10^{-6}s^2 + 9.49 \times 10^{-5}s + 5.26 \times 10^{-5}}{s}$$
$$G_{r2} = -\frac{0.0283s^2 + 1.5645s + 0.8676}{s}$$
$$G_{r3} = -\frac{0.0056s^2 + 0.3085s + 0.1711}{s}$$

Note that if plant models in all working points are equal, in this case we obtain $G_{ri} = 0$, $i = 1, 2, \ldots, p$. If some of $G_{ri} \approx 0$ it indicates that some parameters of plant model are close to other ones.

Using (4.1) and control algorithm

$$u = F(\theta)\,(Cx - w) + F_d(\theta) C_d \dot{x} \tag{4.29}$$

one obtains the structure for simulation of the closed-loop system with gain scheduled PID controller.

Simulation results (*Figs. 4.2, 4.3*) confirm that *Theorem 4.1* holds. *Fig. 4.2* demonstrates that with the gain-scheduled controller we have obtained practically identical behaviour for closed-loop system even if α changes as shown in *Fig. 4.3*. In figures, $y(t)$ is the output signal, $w(t)$ is the setpoint, $u(t)$ is the controller output, $\alpha(t)$ is exogenous signal on which the system depends and θ is the gain scheduled parameter.

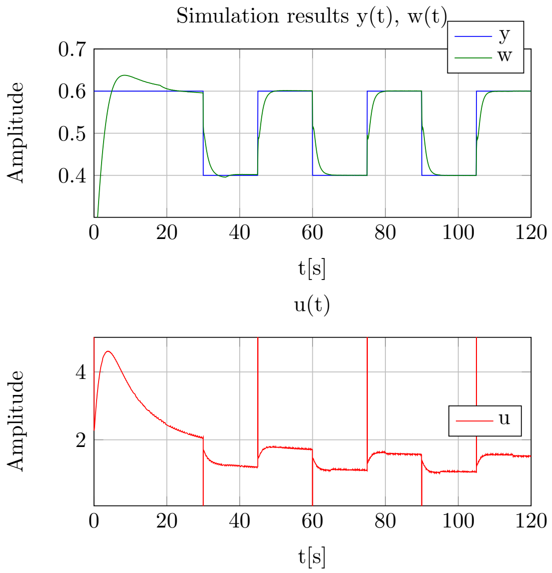

FIGURE 4.2: Simulation results

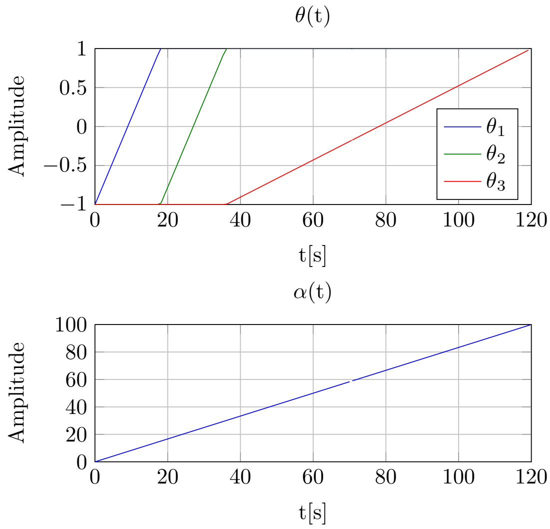

FIGURE 4.3: $\theta(t), \alpha(t)$

The second example is taken from paper [8]. The model in the form (4.1) is extended for gain scheduled PID controller design. The extended model is given as follows for $\theta_1 \in \langle -1, 1 \rangle$

$$A_0 = \begin{bmatrix} -4 & 3 & 5 & 0 \\ 0 & 7 & -5 & 0 \\ 0.1 & -2 & -3 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & -5 & 0 \\ 2 & 5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 \\ 16 \\ 10 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -5 \\ 3.5 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using *Lemma 4.3* we have obtained the gain-scheduled controller in the form (4.4) which after small manipulation can be transformed to the form

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 \tag{4.30}$$

where

$$G_{r0} = \frac{0,139s^2 + 2,0381s + 0,2401}{s}$$
$$G_{r1} = -\frac{0,0027s^2 + 0,014s + 0,004}{s}$$

The simulation results (*Figs. 4.4, 4.5, 4.6, 4.7*) confirm, that *Lemma 4.3* holds. *Figs. 4.4, 4.5, 4.6* demonstrate that with the gain-scheduled controller designed using *Lemma 4.3* we are able to stabilize and control system with such parameter changes. We can see in *Fig. 4.4* that at $\theta = 1$ the system is slow, and the controller output is positive

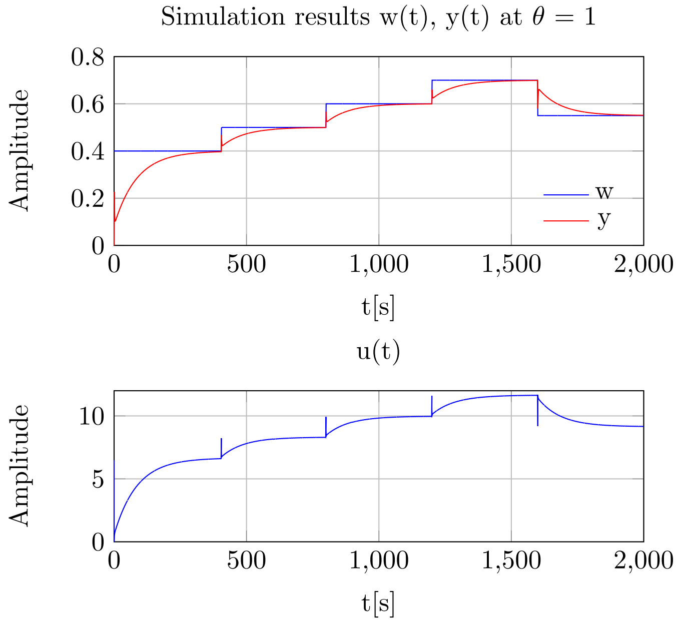

FIGURE 4.4: Simulation results for $\theta = 1$

although when $\theta = 0$ or $\theta = -1$ the system is rapidly fast and the controller output is negative as shown in *Figs. 4.5*, *4.6*. *Fig. 4.7* shows a case, when θ is changing linearly in interval $\langle -1, 1 \rangle$.

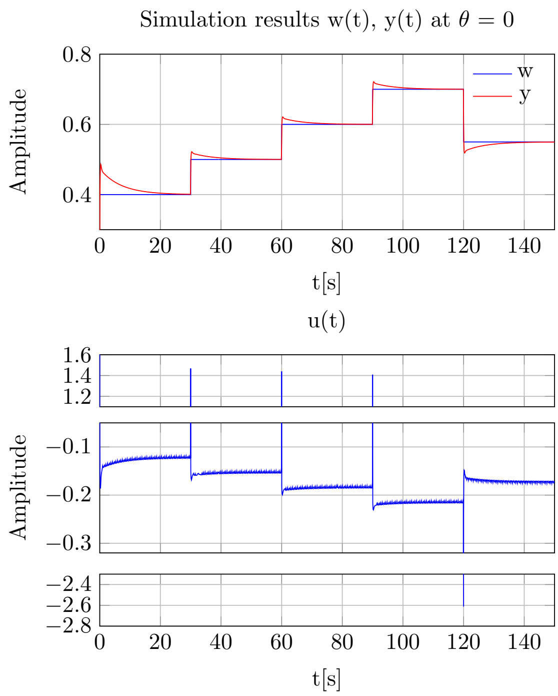

FIGURE 4.5: Simulation results for $\theta = 0$

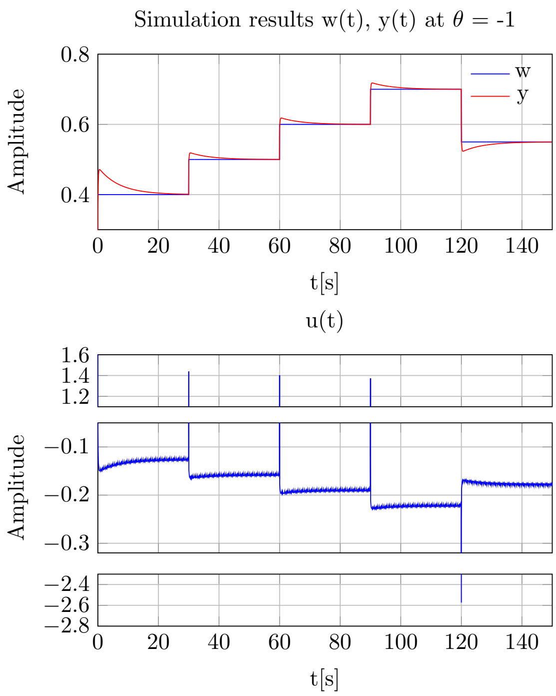

FIGURE 4.6: Simulation results for $\theta = -1$

The third example is a realistic model from Humusoft (magnetic levitation, for more detail see www.humusoft.com). The model consists of a coil and a steel ball levitating in a magnetic field. Position of the steel ball is affected by the intensity of the magnetic

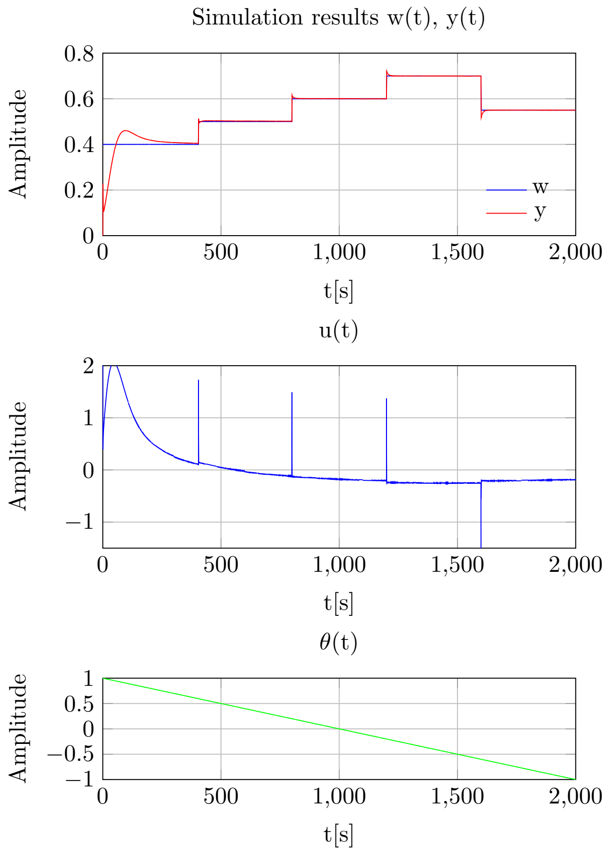

FIGURE 4.7: Simulation results for $\theta \in \langle -1, 1 \rangle$

field and is measured by a linear induction sensor connected to A / D converter. In terms of system theory it is an unstable nonlinear dynamic system with one input (amplifier voltage for coil) and one output (ball position).

We split the ball position (voltage converted by the data acquisition card and scaled to 0÷1 machine unit [MU]) into 3 operating points

1. Position: *0.3 MU* $\longrightarrow \quad \theta_1 = -1, \quad \theta_2 = -1$
2. Position: *0.5 MU* $\longrightarrow \quad \theta_1 = +1, \quad \theta_2 = -1$
3. Position: *0.7 MU* $\longrightarrow \quad \theta_1 = +1, \quad \theta_2 = +1$

In these working points identified[1] plant transfer functions are

$$
\begin{aligned}
G_{s1} &= \frac{2.0921}{0.000264s^2 + 0.0004s - 1} \\
G_{s2} &= \frac{2.2487}{0.00027s^2 + 0.0032s - 1} \\
G_{s3} &= \frac{2.1205}{0.000155s^2 + 0.0065s - 1}
\end{aligned}
\tag{4.31}
$$

[1]Transfer functions were identified in closed-loop system

The above transfer functions are transformed to the time domain to obtain the scheduling model in the form (4.1). The obtained model is extended for the gain-scheduled PID controller design. The extended model is given as follows

$$A_0 = \begin{bmatrix} 0 & 4.1667\times10^3 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5.5 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 833.3333 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$B_0 = \begin{bmatrix} 8.7083\times10^3 \\ 0 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} -805 \\ 0 \\ 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 2.7667\times10^3 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using *Lemma 4.3* with weighting matrices $R = rI$, $r = 1$, $Q = qI$, $q = 1\times10^{-1}$, $S = sI$, $s = 1\times10^{-3}$ (when increasing q or s with respect to r in the first case dynamic behaviour of the closed-loop system becomes faster and in the second case the overshoot of closed-loop system is smaller, for more detail see [17]) we have obtained gain scheduled controller in the form (4.4) which after small manipulation can be transformed to the form

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 \tag{4.32}$$

where

$$G_{r0} = \frac{0.0926s^2 + 2.2966s + 1.7304}{s}$$
$$G_{r1} = -\frac{0.02s^2 + 0.0103s - 0.0007}{s}$$
$$G_{r2} = \frac{0.0017s^2 - 0.0009s + 0.0016}{s}$$

Simulation results are shown in *Fig. 4.8*, where $y(t)$ is the output signal, $w(t)$ is the setpoint, $u(t)$ is the controller output and θ_1, θ_2 are the scheduled parameters calculated from the output signal.

4.5 Conclusion

The paper addresses the problem of the gain-scheduled controller design which ensures the closed-loop stability and guaranteed cost for all scheduled parameter changes. The proposed procedure is based on the Lyapunov theory of stability, guaranteed cost and BMI. In the gain-scheduled controller design procedure one can include the maximal value of the rate of gain-scheduled parameter changes, which allows to decrease conservativeness and obtain the controller with a given performance. The obtained simulation results show that the gain-scheduled controller may give a better performance of closed-loop system for all changes of scheduled parameter than a classical one including robust controller. Another advantage of this method is the fact that we can affect the quality and cost with weighting matrices R, Q, S. Numerical examples illustrate the effectiveness of the proposed approach.

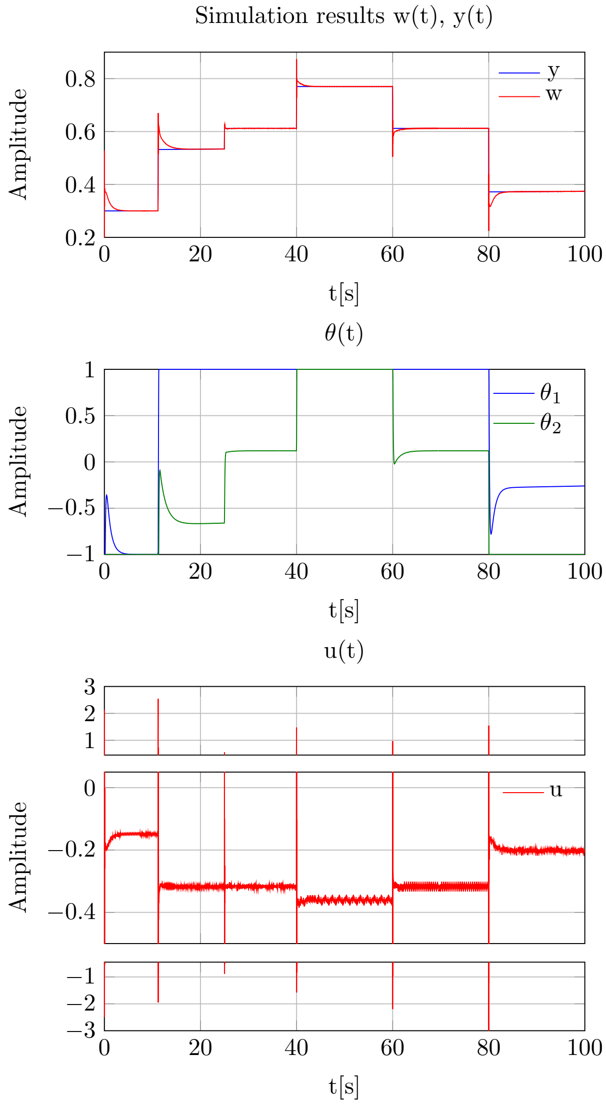

FIGURE 4.8: Simulation results for $R = 1, Q = 1 \cdot 10^{-1}, S = 1 \cdot 10^{-3}$

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5

Gain-Scheduled Controller Design: Variable Weighting Approach (Paper 2)

Abstract

Among the most popular approaches to nonlinear control is gain-scheduled (GS) controller, which can have better performance than robust and other ones. Our approach is based on a consideration that in linear parameter varying (LPV) system, scheduling parameters and their derivatives with respect to time are supposed to lie in a priori given hyper rectangles. To access the performance quality a new quadratic cost function is used, where weighting matrices are time varying depends on scheduled parameter. The class of control structure includes decentralised fixed order output feedbacks like PID controller. Numerical examples illustrate the effectiveness of the proposed approach.

Keywords: Gain-scheduled control, decentralised control, Lyapunov function, quadratic cost function, MIMO LPV systems, PID controller.

5.1 Introduction

Consider a linear parameter varying (LPV) system with state space matrices which are fixed functions of known vector parameter varying $\theta(t)$. This model can be a linear time invariant (LTI) plant model which is result from linearisation of the nonlinear plants along trajectories of the known parameter $\theta(t) \in \langle \underline{\theta}, \overline{\theta} \rangle$. In this note the following LPV system will be used

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u \\ y &= Cx \end{aligned} \tag{5.1}$$

where

$$A(\theta(t)) = A_0 + A_1\theta_1(t) + \ldots + A_p\theta_p(t)$$
$$B(\theta(t)) = B_0 + B_1\theta_1(t) + \ldots + B_p\theta_p(t)$$

and $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control input, $y \in \mathbb{R}^l$ is the measurement output vector, A_0, B_0, A_i, B_i, $i = 1, 2, \ldots, p$, C are constant matrices of appropriate dimension, $\theta(t) \in \langle \underline{\theta}, \overline{\theta} \rangle \in \Omega$ vector of time-varying plant parameters.

The main motivation for our work lies in [1–5]. In the paper [1] the author tackles the design problem of gain scheduled controllers for LPV systems via parameter-dependent Lyapunov function. Recently, [2] proposed the design method for gain scheduled problem using a similar technique to [1]. Improved stability analysis and gain scheduled controller synthesis for parameter-dependent systems are proposed in [3]. Survey of scheduled controller analysis and synthesis are presented in papers [4] and [5].

In this note our approach is based on

- A consideration of the LPV systems (5.1), scheduling parameters $\theta_i, i = 1, 2, \ldots, p$ and their derivatives with respect to time are supposed to lie in a priori given hyper rectangles, $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$.
- Affine quadratic stability (AQS) introduced by [6].
- We use the notion of guaranteed cost to guarantee the performance of closed-loop system.
- The class of control structure includes decentralised fixed order output feedback like PID controller.

The paper is organised as follows. *Section 5.2* brings preliminaries and problem formulation. The main result is presented in *Section 5.3*. In *Section 5.4*, numerical example illustrate the effectiveness of the proposed approach.

5.2 Preliminaries and Problem Formulation

Consider an LPV system with p independent scheduling parameters in the form (5.1). The output feedback control law is considered for PID controller in the form

$$u(t) = F(\theta)y + F_d(\theta)\dot{y} = F(\theta)Cx + F_dC_d\dot{x} \tag{5.2}$$

where

$$F(\theta) = F_0 + \sum_{i=1}^{p} F_i\theta_i$$

is a static output feedback gain scheduled matrix for PI controller and

$$F_d(\theta) = F_{d_0} + \sum_{i=1}^{p} F_{d_i}\theta_i$$

is a static output feedback gain scheduled matrix for D part of controller. Substituting (5.2) to (5.1) and after some manipulation we can obtain the closed-loop system in the following form

$$A_d(\theta)\dot{x} = A_c(\theta)x \tag{5.3}$$

where

$$A_d(\theta) = I - B(\theta)F_d(\theta)C_d$$
$$A_c(\theta) = A(\theta) + B(\theta)F(\theta)C$$

To access the performance quality a quadratic cost function [7] known from LQ theory is often used. In this note the original quadratic cost function is used, where weighting matrices depends on scheduling parameters. Using this approach we can affect on performance quality in each working point separately. The quadratic cost function is in the form

$$J(\theta) = \int_0^{\infty} \left(x^T Q(\theta)x + u^T Ru + \dot{x}^T S(\theta)\dot{x}\right) dt \tag{5.4}$$

where

$$Q(\theta) = Q_0 + \sum_{i=1}^{p} Q_i\theta_i, \quad Q_i = Q_i^T \geq 0$$
$$S(\theta) = S_0 + \sum_{i=1}^{p} S_i\theta_i, \quad S_i = S_i^T \geq 0$$

and $R > 0$. The guaranteed cost is defined in a standard way.

Definition 5.1. Consider the system (5.1) with control algorithm (5.2). If there exists a control law u^* and a positive scalar J^* such that the closed-loop system (5.3) is stable and the value of closed-loop cost function (5.4) satisfies $J \leq J^*$ then J^* is said to be a guaranteed cost and u^* is said to be guaranteed cost control law for system (5.1).

Definition 5.2. [8] The linear closed-loop system (5.3) for $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ is affinally quadratically stable if and only if there exist $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that

$$P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i > 0 \tag{5.5}$$

and for the first derivative of Lyapunov function $V(\theta) = x^T P(\theta)x$ along the trajectory of closed-loop system (5.3) holds

$$\frac{dV(x,\theta)}{dt} = x^T V_v(\theta)x < 0 \tag{5.6}$$

where

$$\begin{aligned} V_v(\theta) &= A_{cd}(\theta)^T P(\theta) + P(\theta)A_{cd}(\theta) + \frac{dP(\theta)}{dt} \\ \frac{dP(\theta)}{dt} &= \sum_{i=1}^{p} P_i\dot{\theta}_i \leq \sum_{i=1}^{p} P_i\rho_i \\ A_{cd}(\theta) &= A_d(\theta)^{-1}A_c(\theta) \end{aligned}$$

From LQ theory we introduce the well known results.

Lemma 5.1. *Consider the closed-loop system (5.3). Closed-loop system (5.3) is affinally quadratically stable with guaranteed cost if and only if the following inequality holds*

$$B_e = \min_u \left\{ \frac{dV(\theta)}{dt} + x^T Q(\theta)x + u^T Ru + \dot{x}^T S(\theta)\dot{x} \right\} \leq 0 \tag{5.7}$$

for all $\theta \in \Omega$ *and* $\dot{\theta} \in \Omega_t$

5.3 Main Results

In this section we presented the gain scheduled controller design procedure which guarantees the affine quadratic stability and required guaranteed costs for all $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$. The main results for the case of gain scheduled closed-loop stability analysis reduces to LMI condition and for gain scheduled controller synthesis to BMI one.

The main results of this section is given by following theorem

Theorem 5.1. *Closed-loop system (5.3) is AQS if there exists* $p+1$ *symmetric matrices* $P_0, P_1, \ldots, P_p$*, satisfying (5.5), matrices* N_1 *and* N_2 *and gain scheduled matrices* $F(\theta)$ *and* $F_d(\theta)$ *satisfying.*

$$\begin{aligned} M_{ij} + M_{ji} &< 0 \\ i &= 1, 2, \ldots, p \\ j &= 1, 2, \ldots, p \end{aligned} \tag{5.8}$$

where

$$M_{ij} = \begin{bmatrix} W_{11}^{ij} & W_{12}^{ij} \\ W_{12}^{ij\,T} & W_{22}^{ij} \end{bmatrix} \tag{5.9}$$

$$
\begin{aligned}
W_{11}^{ij} &= N_1 A_d^{ij} + (A_d^{ij})^T N_1 + \frac{S_0}{\theta_m^2} + \frac{1}{\theta_m} S_i + C_d^T F_d^{ij} C_d \\
W_{12}^{ij} &= -N_1 A_c^{ij} + (A_d^{ij})^T N_2^T + \frac{P_0}{\theta_m^2} + \frac{1}{\theta_m} P_i \\
&\quad + C_d^T F_d^{ij} C \\
W_{22}^{ij} &= -N_2 A_c^{ij} - (A_c^{ij})^T N_2^T + \frac{1}{\theta_m^2} \left(\sum_{k=1}^{p} P_k \rho_k + Q_0 \right) \\
&\quad + \frac{1}{\theta_m} Q_i + C^T F_p^{ij} C \\
A_d^{ij} &= \frac{1}{\theta_m^2} I - \left[\frac{1}{\theta_m^2} B_0 F_{d0} + \frac{1}{\theta_m} B_0 F_{di} + \frac{1}{\theta_m} B_i F_{d0} \right. \\
&\quad \left. + B_i F_{dj} \right] C_d \\
F_d^{ij} &= \frac{1}{\theta_m^2} F_{d0}^T R F_{d0} + \frac{1}{\theta_m} \left(F_{d0} R F_{di} + F_{di}^T \right. \\
&\quad \left. + F_{di} R F_{d0} \right) + F_{di}^T R F_{dj} \\
\theta_m &= \sum_{i=1}^{p} \theta_i \\
A_c^{ij} &= \frac{1}{\theta_m^2} \left(A_0 + B_0 F_0 C \right) + \frac{1}{\theta_m} \left(A_i + B_0 F_i C \right. \\
&\quad \left. + B_i F_0 C \right) + B_i F_j C \\
F_{dr}^{ij} &= \frac{1}{\theta_m^2} F_{d0}^T R F_0 + \frac{1}{\theta_m} \left(F_{d0}^T R F_i + F_{di} R F_0 \right) \\
&\quad + F_{di} R F_j \\
F_p^{ij} &= \frac{1}{\theta_m^2} F_0^T R F_0 + \frac{1}{\theta_m} \left(F_0^T R F_i + F_i R F_0 \right) \\
&\quad + F_i R F_j
\end{aligned}
$$

Proof. Proof is based on the *Lemma 5.1*. Time derivative of Lyapunov function when we using free matrix weighting approach is in the form

$$
\frac{dV}{dt} = z^T \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} z \tag{5.10}
$$

where

$$
\begin{aligned}
Z_{11} &= N_1 A_d(\theta) + A_d^T(\theta) N_1^T \\
Z_{12} &= -N_1 A_c(\theta) + A_d^T(\theta) N_2^T + P(\theta) \\
Z_{21} &= -A_c^T(\theta) N_1^T + N_2 A_d(\theta) + P(\theta) \\
Z_{22} &= -N_2 A_c(\theta) A_d^T(\theta) N_2^T + \sum_{k=1}^{p} P_k \rho_k
\end{aligned}
$$

where $N_1, N_2 \in \mathbb{R}^{n \times n}$ are auxiliary matrices.

When one substitutes to the third part of (5.7) control algorithm (5.2) and the obtained results with (5.3) to (5.7) after some manipulation we obtain (5.9). The proof is completed. □

5.4 Example

An illustrative example is taken from [9]. Consider a simple nonlinear plant with parameter varying coefficients

$$\begin{aligned} \dot{x}(t) &= a(\alpha)x(t) + b(\alpha)u(t) \\ y(t) &= x(t) \end{aligned} \tag{5.11}$$

where $\alpha(t) \in \mathbb{R}$ is an exogenous signal that changes the parameters of the plant as follows

$$a(\alpha) = -6 - \frac{2}{\pi}\arctan\left(\frac{\alpha}{20}\right) \tag{5.12}$$

$$b(\alpha) = \frac{1}{2} + \frac{5}{\pi}\arctan\left(\frac{\alpha}{20}\right) \tag{5.13}$$

Let the aim is to design gain-scheduled PID controller which will guarantee the closed-loop stability and guaranteed cost for $\alpha \in \langle 0, 100\rangle$. We will demonstrate that with our gain-scheduled controller design we can obtain for closed-loop system practically identical behaviour for each working point. To be able to demonstrate this feature, let us divide the working area to 2 sections (with 3 working points) so that in each area where the plant parameter changes they are nearly linear (*Fig. 5.1* – the green lines indicates the chosen working points).

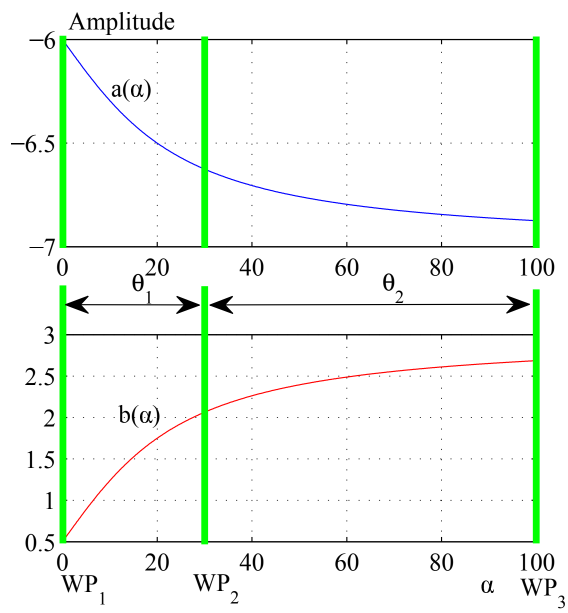

FIGURE 5.1: Exogenous signal $\alpha(t)$

In these working points calculated transfer functions are:

$$\begin{aligned} G_{s1}|_{\alpha=0} &= \frac{0.5}{s+6}, \quad G_{s2}|_{\alpha=30} = \frac{2.064}{s+6.626} \\ G_{s3}|_{\alpha=100} &= \frac{2.686}{s+6.874}, \end{aligned} \tag{5.14}$$

Above transfer functions we transform to time domain to obtain scheduling model in the form (5.1). The obtained model we extended for gain scheduled PID controller design. The extended model is given as follows

$$\begin{aligned} A_0 &= \begin{bmatrix} -6.4370 & 0 \\ 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -0.3130 & 0 \\ 0 & 0 \end{bmatrix} \\ A_2 &= \begin{bmatrix} -0.1240 & 0 \\ 0 & 0 \end{bmatrix} \\ B_0 &= \begin{bmatrix} 1.5930 \\ 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.7820 \\ 0 \end{bmatrix} \\ B_2 &= \begin{bmatrix} 0.3110 \\ 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = 0 \end{aligned}$$

Using *Theorem 5.1* with weighting matrices $Q_i = q_i I$, $q_1 = q_2 = q_3 = 1 \times 10^{-4}$, $R = rI$, $r = 1$, $S_i = s_i I$, $s_1 = s_2 = s_3 = 1 \times 10^{-7}$ we obtain gain scheduled controller in the form

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 \tag{5.15}$$

where

$$\begin{aligned} G_{r0} &= \frac{0.3033s^2 + 2.3036s + 2.0949}{s} \\ G_{r1} &= -\frac{3.93 \times 10^{-6}s^2 + 8.86 \times 10^{-5}s + 3.13 \times 10^{-5}}{s} \\ G_{r2} &= -\frac{0.0724s^2 + 1.6323s + 0.5773}{s} \end{aligned}$$

Simulation results (*Figs. 5.2, 5.3*) confirm, that *Theorem 5.1* holds, but we can see also that with equal q_i, s_i we don't obtain identical behaviour in each working point.

We can change the weighting matrices in the 1. working point to get required performance quality. An another gain-scheduled controller was obtained with weighting matrices $Q_i = q_i I$, $q_1 = 1 \times 10^{-2}$, $q_2 = q_3 = 1 \times 10^{-4}$ $R = rI$, $r = 1$, $S_i = s_i I$, $s_1 = s_2 = s_3 = 1 \times 10^{-7}$

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 \tag{5.16}$$

where

$$G_{r0} = \frac{0.5554s^2 + 0.8513s + 2.7286}{s}$$
$$G_{r1} = -\frac{0.0064s^2 + 0.0559s + 0.0653}{s}$$
$$G_{r2} = -\frac{0.0589s^2 + 0.5173s + 0.6048}{s}$$

Simulation results (*Figs. 5.4, 5.5*) confirm, that with variable weighting matrices we can affect performance quality separately in each working points and we can tune the system to the desired conditions.

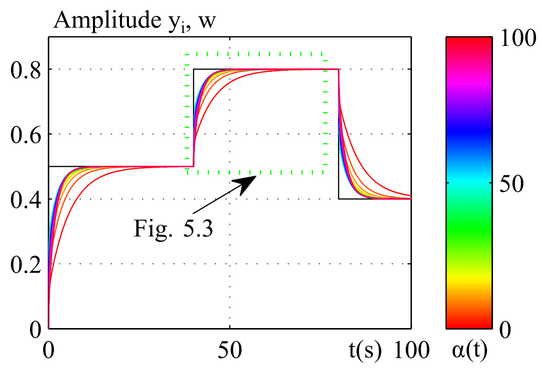

FIGURE 5.2: Simulation results with GSC (5.15)

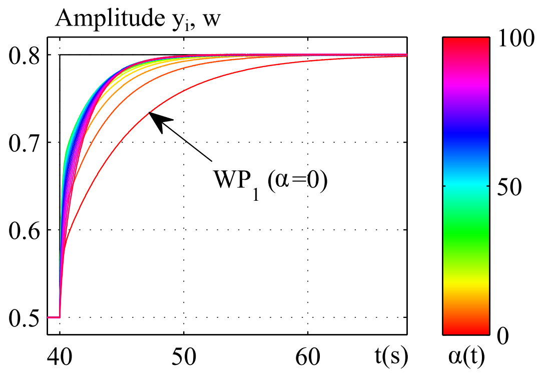

FIGURE 5.3: Simulation results with GSC (5.15) – zoomed

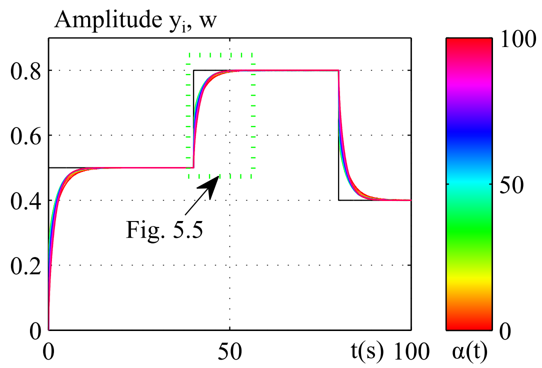

FIGURE 5.4: Simulation results with GSC (5.16)

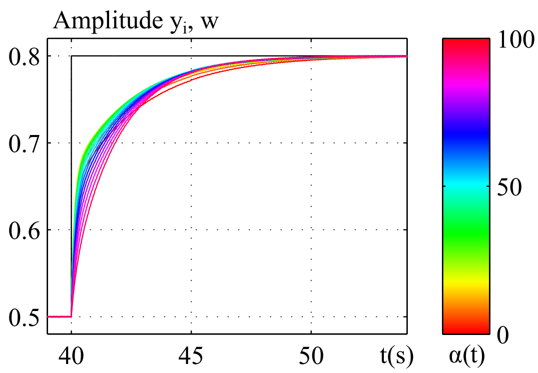

FIGURE 5.5: Simulation results with GSC (5.16) – zoomed

With our gain-scheduled controller design approach we can tune also the change of states with weighting matrices S_i and we can influence to the overshot and oscillation and make the system more slowly.

Let the system to be more slowly in the last working point (WP_3: $\alpha = 100$). An another gain-scheduled controller was obtained with weighting matrices $Q_i = q_iI$, $q_1 = 1 \times 10^{-2}$, $q_2 = q_3 = 1 \times 10^{-4}$, $R = rI$, $r = 1$, $S_i = s_iI$, $s_1 = s_2 = 1 \times 10^{-7}$, $s_3 = 1 \times 10^{-1}$

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 \tag{5.17}$$

where

$$G_{r0} = \frac{0.2161s^2 - 0.1509s + 0.7893}{s}$$
$$G_{r1} = -\frac{0.0095s^2 + 0.1084s + 0.3770}{s}$$
$$G_{r2} = -\frac{0.0088s^2 + 0.1010s + 0.3515}{s}$$

Simulation results are shown in *Figs. 5.6*, *5.7*. Gain-scheduled controller obtained with our controller design method is remains stable under slowly parameter changes too. This is shown in *Figs. 5.8*, *5.9* with gain-scheduled controller (5.16).

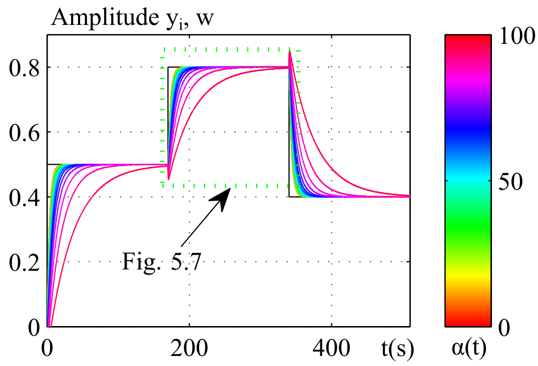

FIGURE 5.6: Simulation results with GSC (5.17)

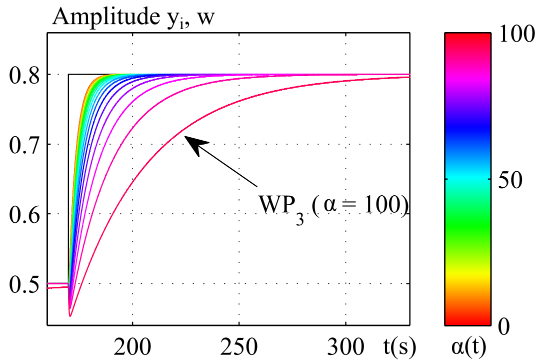

FIGURE 5.7: Simulation results with GSC (5.17) – zoomed

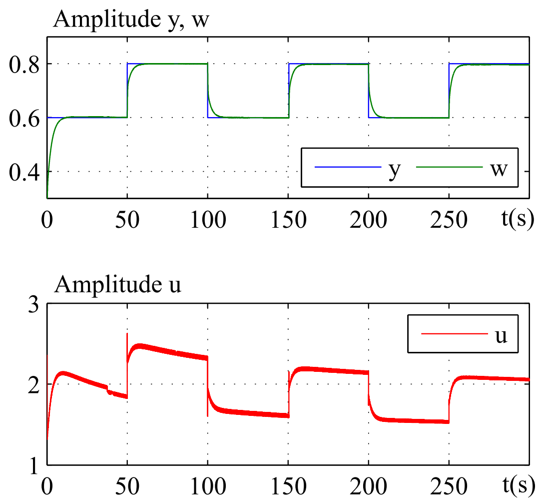

FIGURE 5.8: Simulation results $(y(t), w(t), u(t))$ with GSC (5.16)

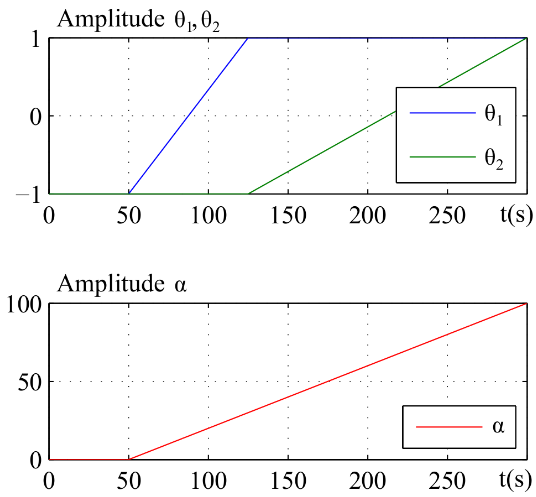

FIGURE 5.9: Simulation results $(\theta(t), \alpha(t))$ with GSC (5.16)

5.5 Conclusion

This paper addresses the problem to design gain-scheduled controller which guarantee the closed-loop stability and performance for all scheduled parameter changes. The proposed original procedure is based on Lyapunov theory of stability, notion of guaranteed cost and BMI. Using original variable weighting matrices we can affect performance quality separately in each working points and we can tune the system to the desired condition through all parameter changes. Numerical example illustrate the effectiveness of the proposed approach.

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6

Design of Robust Gain-Scheduled PI Controllers (Paper 3)

Abstract

A novel approach to robust gain-scheduled controller design is presented. The proposed design procedure is based on the robust stability condition developed for an uncertain LPV system model introduced in this paper. The feasible design procedures are obtained in the form of BMI or LMI. The obtained design results and their properties are illustrated on simulation examples.

Keywords: Gain-scheduled controller, robust controller, parameter-dependent Lyapunov function, quadratic gain-scheduled cost function, LPV systems.

6.1 Introduction

Gain scheduling belongs to the most popular approaches to linear parameter varying systems (LPV) control design. However, in the absence of corresponding results, this design provides no guarantees of robust stability, performance or even nominal stability of the overall gain-scheduled control [1]. In many applications, a controller must accommodate a plant with changing dynamics, where the dynamics is strongly dependent on the operating conditions. Developing a nonlinear plant model, the gain-scheduled controller can be designed by the standard approach described in [2], [3] and [4] using linear controller design techniques. In such cases, the designed gain-scheduled controller must be able to stabilize and guarantee a reasonable performance for all operating conditions. The question remains, what happens with a closed-loop system if the developed physical nonlinear model is not precise enough? In such a case, frequent in applications, there is a need for robust controller to cope with model uncertainty. Various robust controller design methods for gain-scheduled plant are available in literature. In [5] the authors consider the case where uncertainties of LPV system are modeled by diagonal matrix with norm bounded unknown parameters. The control objectives are internal stability and disturbance attenuation in the case of a bounded induced L_2 norm. The authors in [6], address the gain scheduling of separately designed controllers to form a robust linear parameter varying controller for an LPV plant. That is, the intent is to integrate a set of separately designed local controllers into a gain-scheduled controller with maximal quadratic H_∞ performance using the LMI approach. A robust PID controller is designed in [7] for the condensing boiler problem. The main feature of the proposed method is that the stability, robustness margins and some performance specifications are guaranteed by linear constraints in the Nyquist diagram. The condensing boiler is described by first order models with time delay. A design of gain-scheduled PI controller is presented in [1] considering that the system uncertainty is assumed to be the difference between the nonlinear model (assuming to be an exact model) and the nominal linear model. Additionally, the input constraints problem is explicitly addressed in this paper.In the article [8] presents the central finite-dimensional H_∞ controller for linear time-varying systems with unknown parameters, that is suboptimal for a given threshold with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. The paper [9] is concerned with the problem of the robust H_∞ filtering design for singular linear parameter varying systems with time variant delays. The obtained results are proposed in terms of linear matrix inequality. In [10], the problem of attenuation of sinusoidal disturbances with uncertain and arbitrarily time-varying frequencies is solved by synthesis of LPV controllers using the L_2 - gain method. A robust gain-scheduled design procedure based on L_2 controller synthesis method is proposed for LPV systems in [11].The gap matrix approach to the design of a robust gain-scheduled controller for LPV systems is studied in [12]. In [13], quadratic and bi-quadratic (affine) stability approaches are used to design the gain-scheduled controller for each vertex of the plant uncertainty box and the stability of the closed-loop system is verified by LMI. In [14], a commercially available and vertically designed rotor bearing system is modeled and controlled using an LMI and H_∞ based gain-scheduled controller. The paper[15] addresses the problem to gain scheduled controller design when scheduling parameters are belong to the bounded uncertainties. The contribution of [16] is to use standard results motivated by Youla-Kucera parametrization to propose a controller structure and design approach that allows the gain scheduling of linear system such that a robustly stable nonlinear control system is achieved. In [17] provides a novel approach to design continuous time varying H_∞ gain-scheduled worst-case controllers for nonlinear stochastic systems subject to partially known transition jump rates and actuator saturation. The above short survey motivated us to study the following research problem which has not been sufficiently solved yet. Design the robust gain-scheduled controller which will guarantee:

- stability and robustness properties of the closed-loop system when the uncertain plant parameters Π for all scheduled parameters $\theta \in \Omega_s$ and their rate $\dot{\theta} \in \Omega_t$ lie in the given polytopic (convex) uncertainty box Ω, that is $\Pi \in \Omega$, $\theta \in \Omega_s$ and $\dot{\theta} \in \Omega_t$;

- performance (guaranteed cost) for closed-loop system for all $\Pi \in \Omega$, $\theta \in \Omega_s$ and $\dot{\theta} \in \Omega_t$;
- parameter dependent quadratic stability.

In this paper, we provide, to the authors' best knowledge, an alternative novel approach to the robust gain-scheduled control problem. The proposed LPV uncertainty model, introduced in *Section 6.2*, is used to formulate a robust gain scheduling control problem. The main result is presented in *Section 6.3*. The design procedure is based on the new developed robust stability condition for the gain-scheduled controller. In *Section 6.4* the LMI gain-scheduled controller design is presented and in *Section 6.5*, numerical examples illustrate the effectiveness of the proposed approaches.

6.2 Problem formulation and preliminaries

Consider a linear continuous time parameter varying (LPV) uncertain system

$$\begin{aligned} \dot{x} &= \bar{A}\left(\xi,\theta\right)x + \bar{B}\left(\xi,\theta\right)u \\ y &= Cx \end{aligned} \tag{6.1}$$

where parameter varying system matrices are affine with respect to scheduled parameter θ

$$\begin{aligned} \bar{A}\left(\xi,\theta\right) &= A_0\left(\xi\right) + \sum_{i=1}^{s} A_i\left(\xi\right)\theta_i \quad &\in \mathbb{R}^{n\times n} \\ \bar{B}\left(\xi,\theta\right) &= B_0\left(\xi\right) + \sum_{i=1}^{s} B_i\left(\xi\right)\theta_i \quad &\in \mathbb{R}^{n\times m} \end{aligned} \tag{6.2}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ denote the state, control input and controlled output, respectively. Matrices $A_i\left(\xi\right)$, $B_i\left(\xi\right)$, $i = 0, 1, \ldots s$ belong to the convex and bounded set: a polytope with N vertices that can be formally defined as

$$\begin{aligned} \Omega :=&\{A_i(\xi) \in \mathbb{R}^{n\times n}, B_i(\xi) \in \mathbb{R}^{n\times m} : \{A_i(\xi), B_i(\xi)\} := \\ &\sum_{j=1}^{N} \left(A_{ij}, B_{ij}\right)\xi_j, \sum_{j=1}^{N} \xi_j = 1, \xi_j \geq 0\} \end{aligned} \tag{6.3}$$

where s is the number of scheduled parameters; ξ_j, $j = 1, 2, \ldots, N$ are constant but unknown parameters respective to uncertainties in system matrices $A_i(\xi)$, $B_i(\xi)$; A_{ij}, B_{ij}, C are constant matrices of corresponding dimensions respective to uncertainty polytope vertices; $\theta \in \mathbb{R}^s$ is a vector of known constant or possible time-varying real gain scheduling parameters. We assume that both lower and upper bounds are available for these parameters' value and variation rates. Specifically

- Each parameter θ_i, $i = 1, 2, \ldots, s$ ranges between known extremal values

$$\theta_i \in \Omega_s := \left\{\theta_i \in \left\langle \underline{\theta}_i, \overline{\theta}_i \right\rangle, i = 1, 2, \cdots s\right\}. \tag{6.4}$$

- The rate of variation $\dot{\theta}_i$ is well defined at all times and satisfies

$$\dot{\theta}_i \in \Omega_t := \left\{\dot{\theta}_i \in \left\langle \underline{\dot{\theta}}_i, \overline{\dot{\theta}}_i \right\rangle, i = 1, 2, \ldots s\right\}. \tag{6.5}$$

Note that system (6.1) consists of two type vertices. The first one is due to the gain-scheduled parameter θ with $T = 2^s$ vertices, θ – vertices and second set of vertices are due to the system uncertainties, N vertices – ξ vertices. For the gain-scheduled I part controller design the states of system (6.1) need to be extended in such a way that the static output feedback control algorithm can provide proportional (P) and integral (I) parts of the designed PI robust controller. For more detail see [18]. Assume that system (6.1) allows to design the PI controller with static output feedback. The following problem is studied in this paper for the class of uncertain LPV systems (6.1).

Problem 6.1. Design a robust output feedback gain-scheduled PI controller with control algorithm

$$u = \left(F_0 + \sum_{i=1}^{s} F_i\theta_i\right) y = F(\theta)Cx \tag{6.6}$$

such that the controller guarantees robust parameter dependent quadratic stability and guaranteed cost with respect to the performance index (6.8) for the respective closed-loop system (6.7) with convex uncertainty domain (6.3)

$$\dot{x} = \left(\bar{A}(\xi,\theta) + \bar{B}(\xi,\theta)F(\theta)C\right) x = A_c(\xi,\theta)x \tag{6.7}$$

To assess the performance quality, a quadratic cost function known from LQ theory is used

$$J_c = \int_0^\infty \left(x^T Q x + u^T R u\right) dt = \int_0^\infty J(t)dt \tag{6.8}$$

The respective notion of guaranteed cost is given in the next definition.

Definition 6.1. Consider system (6.1) and controller (6.6). If there exist a control law u^* and a positive scalar J^* such that the respective closed-loop system (6.7) is stable and the value of the closed-loop cost function (6.8) satisfies $J_c \leq J^*$, then J^* is said to be guaranteed cost and u^* is said to be the guaranteed cost control law for system (6.1).

Recall the well known result from LQ theory which will be used below to prove one of the main results.

Lemma 6.1. *[19] Consider the system* (6.1) *with control algorithm* (6.6)*. Control algorithm* (6.6) *is the guaranteed cost control law for the closed-loop system* (6.7) *if and* *only if there exists a parameter dependent Lyapunov function* $V(\xi, \theta)$ *such that the following condition holds*

$$B_e(\xi, \theta) = \min_u \left(\frac{dV(\xi, \theta, u)}{dt} + J(t) \right) \leq 0 \tag{6.9}$$

Uncertain system (6.1) with control algorithm (6.6) conforming to *Lemma 6.1* is called robust parameter dependent quadratically stable with guaranteed cost.

We proceed with the notion of multi-convexity of a scalar quadratic function, and its qualities classified in the next lemma, [20].

Lemma 6.2. *Consider a scalar quadratic function of* $\theta \in \mathbb{R}^s$

$$f(\theta) = \alpha_0 + \sum_{i=1}^{s} \alpha_i \theta_i + \sum_{i=1}^{s} \sum_{j>i}^{s} \beta_{ij} \theta_i \theta_j + \sum_{i=1}^{s} \gamma_i \theta_i^2 \tag{6.10}$$

and assume that $f(\cdot)$ *is multiconvex, that is* $\frac{\partial^2 f}{\partial \theta_i^2} = 2\gamma_i \geq 0,\ i = 1, 2, \cdots s$. *Then* $f(\cdot)$ *is negative in the hyper rectangle* (6.4) *if and only if it takes negative values at the corners of* (6.4)*, that is, if and only if* $f(\theta) < 0$ *for all vertices of the set given by* (6.4).

6.3 Main results

This section formulates the theoretical approach to the robust gain-scheduled controller design for polytopic system (6.1) which ensures closed-loop system parameter dependent quadratic stability and guaranteed cost (6.8), for all uncertain plant parameters $\Pi \in \Omega$, gain scheduling parameters $\theta \in \Omega_s$ and $\dot{\theta} \in \Omega_t$. The main result on robust stability for the gain-scheduled control system is given in the next theorem.

Theorem 6.1. *The closed-loop system* (6.7) *is robust parameter dependent quadratically stabilizable with guaranteed cost if there exist a positive definite matrix* $P(\xi, \theta) \in \mathbb{R}^{n \times n}$*, matrices* $N_1, N_2 \in \mathbb{R}^{n \times n}$ *and gain-scheduled controller* (6.6) *such that*
a)

$$\begin{aligned} L(\xi, \theta) = {} & M_0(\xi) + \sum_{i=1}^{s} M_i(\xi) \theta_i \\ & + \sum_{i=1}^{s} \sum_{j>i}^{s} M_{ij}(\xi) \theta_i \theta_j + \sum_{i=1}^{s} M_{ii}(\xi) \theta_i^2 < 0 \end{aligned} \tag{6.11}$$

b)

$$M_{ii}(\xi) \geq 0, \tag{6.12}$$

$$i = 1, 2, \ldots, s \quad \forall \theta \in \Omega_s, \quad \dot{\theta} \in \Omega_t, \quad \sum_{j=1}^{N} \xi_j = 1, \quad \xi_j \geq 0$$

where due to structure of system (6.1) we consider the parameter dependent Lyapunov function in the form

$$P(\xi,\theta) = P_0(\xi) + \sum_{i=1}^{s} P_i(\xi)\theta_i > 0 \tag{6.13}$$

$$\begin{aligned}
M_0(\xi) &= \begin{bmatrix} M_{011} & M_{012} \\ M_{012}^T & M_{022} \end{bmatrix}, \quad M_i(\xi) = \begin{bmatrix} M_{i11} & M_{i12} \\ M_{i12}^T & M_{i22} \end{bmatrix} \\
M_{ii}(\xi) &= \begin{bmatrix} M_{ii11} & M_{ii12} \\ M_{ii12}^T & M_{ii22} \end{bmatrix}, \quad M_{ij}(\xi) = \begin{bmatrix} M_{ij_{11}} & M_{ij_{12}} \\ M_{ij_{12}}^T & M_{ij_{22}} \end{bmatrix} \\
M_{011} &= N_1^T + N_1, \ M_{012} = -N_1^T A_{0c}(\xi) + P_0(\xi) + N_2, \\
M_{022} &= -N_2^T A_{0c}(\xi) - A_{0c}(\xi)^T N_2 \\
&\quad + Q + C^T F_0^T R F_0 C + \sum_{i=1}^{s} P_i(\xi)\dot{\theta}_i \\
A_{0c}(\xi) &= A_0(\xi) + B_0(\xi) F_0 C, \quad M_{i11} = 0 \\
M_{i12} &= -N_1^T A_{ic}(\xi) + P_i(\xi) \\
M_{i22} &= -N_2^T A_{ic}(\xi) - A_{ic}(\xi)^T N_2 \\
&\quad + C^T \left(F_0^T R F_i + F_i^T R F_0\right) C \\
A_{ic}(\xi) &= A_i(\xi) + \left(B_0(\xi) F_i + B_i(\xi) F_0\right) C \\
M_{ii11} &= 0, \ M_{ii12} = -N_1 A_{iic}(\xi) \\
M_{ii22} &= -N_2^T A_{iic}(\xi) - A_{iic}(\xi)^T N_2 + C^T F_i^T R F_i C \\
A_{iic}(\xi) &= B_i(\xi) F_i C, \quad M_{ij_{11}} = 0, \ M_{ij_{12}} = -N_1^T A_{ijc}(\xi) \\
M_{ij_{22}} &= -N_2^T A_{ijc}(\xi) - A_{ijc}(\xi)^T N_2 \\
&\quad + C^T \left(F_i^T R F_j + F_j^T R F_i\right) C \\
A_{ijc}(\xi) &= \left(B_i(\xi) F_j + B_j(\xi) F_i\right) C \quad i \neq j
\end{aligned}$$

Proof. The proof is based on *Lemma 6.1* and *6.2*. For concrete structure of $V(\xi,\theta) = x^T P(\xi,\theta)x$, *Theorem 6.1* reduces to sufficient condition. To prove *Theorem 6.1* it is sufficient to prove that inequalities (6.11) and (6.12) imply that (6.9) holds for parameter dependent Lyapunov function (6.13). The first time derivative of the Lyapunov function $V(\xi,\theta) = x^T P(\xi,\theta)x$ is

$$\frac{dV(\xi,\theta)}{dt} = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix} \begin{bmatrix} 0 & P(\xi,\theta) \\ P(\xi,\theta) & P(\xi,\dot{\theta}) \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{6.14}$$

where

$$P(\xi,\dot{\theta}) = \sum_{i=1}^{s} P_i(\xi)\dot{\theta}_i$$

Let us now substitute (6.14) and $J(t)$ from (6.8) to (6.9) and summarize the left hand side of the obtained inequality with the following equality

$$[N_1\dot{x} + N_2 x]^T [\dot{x} - A_c(\xi,\theta)x] \\ + [\dot{x} - A_c(\xi,\theta)x]^T [N_1\dot{x} + N_2 x] = 0 \tag{6.15}$$

after some manipulation one obtains

$$B_e(\xi,\theta) = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix} \begin{bmatrix} B_{e11} & B_{e12} \\ B_{e21} & B_{e22} \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{6.16}$$

where

$$\begin{aligned} B_{e11} &= N_1^T + N_1 \\ B_{e12} &= -N_1^T A_c(\xi,\theta) + N_2 + P(\xi,\theta) \\ B_{e21} &= P(\xi,\theta) + N_2^T - A_c(\xi,\theta)^T N_1 \\ B_{e22} &= W(\xi,\theta) \\ W(\xi,\theta) &= -N_2^T A_c(\xi,\theta) - A_c(\xi,\theta)^T N_2 + P(\xi,\dot{\theta}) + Q \\ &+ C^T F_0^T R F_0 C + \sum_{i=1}^{s} C^T \left(F_0^T R F_i + F_i^T R F_0\right) C\theta_i \\ &+ \sum_{i=1}^{s}\sum_{j=1}^{s} C^T F_i^T R F_j C \theta_i \theta_j \end{aligned}$$

Now, to show that inequalities (6.11) and (6.12) imply (6.9), it is enough to prove that $B_e(\xi,\theta) \leq 0$ if (6.11) and (6.12) hold. To prove the latter, we use the multiconvexity quality of $B_e(\xi,\theta)$. Recall that in (6.16) $A_c(\xi,\theta)$ can be rewritten as follows

$$\begin{aligned} A_c(\xi,\theta) = A_0(\xi) &+ B_0(\xi) F_0 C \\ &+ \sum_{i=1}^{s} \left[A_i(\xi) + \left(B_0(\xi) F_i + B_i(\xi) F_0\right) C\right] \theta_i \\ &+ \sum_{i=1}^{s}\sum_{j=1}^{s} B_i(\xi) F_j C \theta_i \theta_j \end{aligned} \tag{6.17}$$

Substituting (6.17) to (6.16) and multiplying from the left and right hand sides with non zero vector z one obtains a scalar quadratic function (6.10). Due to *Lemma 6.2*, this scalar quadratic function is multiconvex and negative if (6.11) and (6.12) hold, which proves the sufficient robust stability conditions of *Theorem 6.1*. □

Note that (6.11) and (6.12) are linear with respect to uncertain parameters ξ_j, $j = 1, 2, \ldots, N$ therefore the above inequalities have to hold for all $j = 1, 2, \ldots, N$ respective to uncertainty domain vertices.
Multiplying entries of (6.11) by

$$\frac{\sum_{i=1}^{s} \theta_i}{\theta_m} = 1 \tag{6.18}$$

where $\theta_m \in \langle \underline{\theta_m}, \overline{\theta_m} \rangle$ assuming that $\theta_m > 0$. After small manipulation one obtains

$$\sum_{i=1}^{s} \sum_{j=1}^{s} (M_o(\xi) + M_i(\xi)\theta_m + M_{ij}(\xi)\theta_m^2)\theta_i\theta_j < 0 \tag{6.19}$$

From (6.19) and *Lemma 6.2* the alternative robust stability conditions follow in the next Lemma.

Lemma 6.3. *The closed-loop system (6.7) is robust parameter dependent quadratically stable with guaranteed cost if there exist a positive definite matrix $P(\xi, \theta)$, matrices N_1, N_2 and gain-scheduled controller (6.6) such that for all θ, ξ- vertices the following robust stability conditions hold*

$$\begin{aligned} &2M_o(\xi) + (M_i(\xi) + M_j(\xi))\underline{\theta_m} + (M_{ij}(\xi) + M_{ji}(\xi))\underline{\theta_m^2} < 0 \\ &2M_o(\xi) + (M_i(\xi) + M_j(\xi))\overline{\theta_m} + (M_{ij}(\xi) + M_{ji}(\xi))\overline{\theta_m^2} < 0 \\ &\overline{M}_{ij} = M_{ij}(\xi) + M_{ji}(\xi) \geq 0; i = 1, 2, \ldots, s; j = i, i+1 \ldots, s \end{aligned} \tag{6.20}$$

Note that in (6.20) the number of θ-vertices reduces to $T = 2$, $\theta_m = \underline{\theta_m}$ or $\theta_m = \overline{\theta_m}$. Note that (6.11), (6.12) and (6.20) are linear with respect to uncertain parameters ξ_j, $j = 1, 2, \ldots, N$ therefore the above inequalities have to hold for all $j = 1, 2, \ldots, N$ respective to uncertainty domain vertices.

Remark 6.1. [20] One can reduce the conservatism of *Theorem 6.1* or *Lemma 6.3* by relaxing the multi-convexity requirement (6.12). Specifically, let $\overline{N}_i$, $i = 1, 2, \ldots, s$ be non-negative symmetric matrices and define the augmented function to (6.11)

$$L_u(\xi, \theta) = L(\xi, \theta) + \sum_{i=1}^{s} \overline{N}_i \theta_i^2 \geq L(\xi, \theta) \tag{6.21}$$

Quadratic form $z^T L_u(\xi, \theta) z$ is a multiconvex function of θ if and only if $M_{ii}(\xi) + \overline{N}_i \geq 0$ for $i = 1, 2, \ldots, s$. This leads to an immediate extension of *Theorem 6.1* or *Lemma 6.3* where (6.11) and (6.12) are replaced by

$$\begin{aligned} &L_u(\xi, \theta) < 0 && \forall \quad \theta \in \Omega_s, \ \ \dot{\theta} \in \Omega_t, \\ & && \xi_j, \ \ j = 1, 2, \ldots, N \\ &M_{ii}(\xi) + \overline{N}_i \geq 0 && i = 1, 2, \ldots, s, \\ & && \xi_j, \ \ j = 1, 2, \ldots, N \\ &\overline{N}_i \geq 0 && i = 1, 2, \ldots, s \end{aligned} \tag{6.22}$$

6.4 LMI gain-scheduled robust controller design

In this paragraph the obtained theoretical results aiming at designing a robust gain-scheduled PI controller (6.11 and 6.20) using BMI approach will be transformed to an

LMI design procedure. At first the gain-scheduled plant (6.1) with variable parameter θ will be transformed to a set of equivalent systems with constant parameters. For the LMI design procedure the modified robust elimination lemma [21] and linearization approach will be adopted [22].

6.4.1 Equivalent gain-scheduled system

Substituting $M_o(\xi)$, $M_i(\xi)$, $M_{ij}(\xi)$ to robust stability condition (6.20) after manipulation one obtains the following robust stability conditions

$$W(\xi) = \{w_{ij}(\xi)\}_{2\times 2} < 0, \quad \overline{M}_{ij} \geq 0 \tag{6.23}$$

where

$$\begin{aligned}
w_{11}(\xi) &= 2(N_1^T + N_1)\\
w_{12}(\xi) &= 2(-N_1^T A_{0c}(\xi) + P_0(\xi) + N_2)\\
&\quad - N_1^T(A_{ic}(\xi) + A_{jc}(\xi))\theta_m + (P_i(\xi) + P_j(\xi))\theta_m\\
&\quad - N_1^T(A_{ijc}(\xi) + A_{jic}(\xi))\theta_m^2\\
w_{22}(\xi) &= 2(-N_2^T A_{0c}(\xi) - A_{0c}(\xi)^T N_2)\\
&\quad - N_2^T(A_{ic}(\xi) + A_{jc}(\xi))\theta_m\\
&\quad - (A_{ic}(\xi)^T + A_{jc}(\xi)^T)N_2\theta_m\\
&\quad - N_2^T(A_{ijc}(\xi) + A_{jic}(\xi))\theta_m^2\\
&\quad - (A_{ijc}(\xi)^T + A_{jic}(\xi)^T)N_2\theta_m^2 + Per_{ij}
\end{aligned}$$

for $i = 1, 2, \dots, s$; $j = i, i+1, \dots, s$ and θ, ξ-vertices. Rewrite $w_{12}(\xi)$ as follows

$$\begin{aligned}
w_{12}(\xi) = N_1^T[&A_o(\xi) + B_o(\xi)F_oC\\
&+ (A_i(\xi) + B_o(\xi)F_i + B_i(\xi)F_0)C\\
&+ A_j(\xi) + (B_0(\xi)F_j + B_j(\xi)F_0)C0.5\theta_m]\\
&- N_1^T(B_i(\xi)F_j + B_j(\xi)F_i)C\theta_m^2\\
&+ P_o(\xi) +)0.5(P_i(\xi) + P_j(\xi)) + N_2
\end{aligned} \tag{6.24}$$

From (6.24) we obtain the following set of gain-scheduled equivalent systems

$$\begin{aligned}
A_{ij}^e(\xi) &= A_0(\xi) + 0.5(A_i(\xi) + A_j(\xi))\\
B_{ij}^e(\xi) &= [b1_{ij} \quad b2_{ij} \quad b3_{ij}]
\end{aligned} \tag{6.25}$$

where

$$\begin{aligned}
b1_{ij} &= B_0(\xi) + 0.5(B_i(\xi) + B_j(\xi))\\
b2_{ij} &= (B_0(\xi) + B_j(\xi)\theta_m)0.5\theta_m\\
b3_{ij} &= (B_0(\xi) + B_i(\xi)\theta_m)0.5\theta_m
\end{aligned}$$

Equivalent gain-scheduled feedback matrix

$$(F_{ij}^e)^T = [F_0^T F_i^T F_j^T]$$

and equivalent closed loop system

$$A_{cij}^e(\xi) = A_{ij}^e(\xi) + B_{ij}^e(\xi)F_{ij}^e C \tag{6.26}$$

Equivalent Lyapunov matrix

$$P_{ij}^e(\xi) = P_0(\xi) + 0.5(P_i(\xi) + P_j(\xi)) \tag{6.27}$$

$$i = 1, 2, \ldots, s; j = i, i+1, \ldots, s$$

On the base of equivalent system condition inequalities (6.23) read as follows

$$\begin{aligned} w_{11}(\xi) &= N_1^T + N_1; \quad \overline{M}_{ij} \geq 0 \\ w_{12}(\xi) &= -N_1^T A_{cij}^e(\xi) + N_2 + P_{ij}^e(\xi) \\ w_{22}(\xi) &= -N_2^T A_{cij}^e(\xi) - A_{cij}^e(\xi)^T N_2 + Per_{ij} \end{aligned} \tag{6.28}$$

To make a simple linearization process let us choose the performance in (6.28) as follows

$$\begin{aligned} Per_{ij} =& Q + C^T (F_{ij}^e)^T \bar{R} F_{ij}^e C = \\ & Q + C^T (F_0^T R F_0 + F_i^T R F_i + F_j^T R F_j) C \end{aligned} \tag{6.29}$$

where $\bar{R} = diag\{R\} \in \mathbb{R}^{3m \times 3m}$.

6.4.2 LMI robust controller design procedure

Recall the main results of the robust elimination lemma [21]

Lemma 6.4. *Conditions (6.23) and (6.28) hold if for ξ-vertices there exits $\gamma > 0$ such that for the set of equivalent systems the following conditions hold*

- *Matrix $P_{ijk}^e + N_2^T - \gamma A_{cijk}^e$ is non singular and*
-

$$A_{cijk}^e{}^T P_{ijk}^e + P_{ijk}^e A_{cijk}^e + Per_{ij} < 0$$

$$\begin{aligned} & Per_{ij} - N_2^T A_{cijk}^e - A_{cijk}^e{}^T N_2 + \gamma^2 A_{cijk}^e{}^T A_{cijk}^e \\ & \quad - (P_{ijk}^e + N_2^T)(-2\gamma + 1)^{-1} I (P_{ijk}^e + N_2^T)^T < 0 \end{aligned} \tag{6.30}$$

where $i = 1, 2, \ldots, s$; $j = i, i+1, \ldots, s$; $k = 1, 2, \ldots, N$

Because of sufficient robust stability conditions the use of *Lemma 6.4* may be conservative. For decreasing the solution conservativeness the following LMI robust controller

design procedure is adopted
Gain-scheduled robust controller LMI design procedure.

1. Calculate the gain-scheduled matrices F_0, F_i, $i = 1, 2, \ldots, s$ from three LMI inequalities

$$\begin{aligned}(A^e_{cijk})^T P^e_{ijk} + P^e_{ijk} A^e_{cijk} \\ + Q + C^T (F^e_{ij})^T \bar{R} F^e_{ij} C < 0\end{aligned}$$

$$\begin{aligned}- N_2^T A^e_{cijk} - (A^e_{cijk})^T N_2 \\ + Q + C^T (F^e_{ij})^T \bar{R} F^e_{ij} C + Q_1 < 0\end{aligned} \tag{6.31}$$

$$\overline{M}_{ij} \geq 0 \tag{6.32}$$

   where

$$\begin{aligned}Q_1 = -(P^e_{ijk} + N_2^T)(-2\gamma + 1)^{-1} I (P^e_{ijk} + N_2^T)^T \\ + \gamma^2 (A^e_{cijk})^T A^e_{cijk} \geq 0\end{aligned}$$

   To obtain inequalities (6.31) and (6.32) to the LMI, the standard linearization approach can be used [22], see Appendix. Note that for the iteration procedure we reduce matrix Q_1 (6.31) to a matrix with constant entries and for the first step of the iteration procedure we choose $Q_1 = 0$.

2. When gain-scheduled matrices $F_0, F_1, \ldots, F_s$ are known, check robust stability by (6.23) or (6.28). The above conditions for robust stability analysis reduce to LMI.

3. If the closed loop system is not robust stable ((6.23) or (6.28) are not feasible), increase q_1, $Q_1 = q_1 I > 0$ and repeat the solution from the first step. In the step 2 the original value of matrix Q_1 is used.

4. If there is no solution with proposed algorithm the robust elimination lemma with two step algorithm fails.

Remark 6.2. (*Theorem 6.1*, *Lemma 6.3*), LMI design procedure can be used also for a quadratic stability test, where Lyapunov function matrix (matrices) are either not dependent on parameter ξ or parameter θ or both, as listed below.

1. Quadratic stability with respect to model parameter variation. For this case we have $P(\theta) = P_0 + \sum_{i=1}^{s} P_i \theta_i$. This Lyapunov function should withstand an arbitrarily fast model parameter variation in the convex set (6.3).

2. Quadratic stability with respect to gain-scheduled parameter θ. For this case $P_i \to 0$, $i = 1, 2, \ldots, s$ and the Lyapunov matrix is $P(\xi, \theta) = P_0(\xi)$. This Lyapunov function can withstand arbitrarily fast θ parameter variations.

3. Quadratic stability with respect to both ξ and θ parameters. The Lyapunov matrix for this case is $P(\xi, \theta) = P_0$ and it withstands arbitrarily fast model and gain-scheduled parameter variations.

Remark 6.3. When one takes into account parameter θ variations in (6.11) or (6.22), the respective number of inequality constraints grows exponentially with the number of s. For this case it is preferable to use a more conservative approach, quadratic stability variant of *Theorem 6.1*, *Lemma 6.3* considering *Remark 6.2*.

6.5 Examples

In this section, the robust gain-scheduled controller design procedure described in *Sections 6.3* and *6.4*, based on a solution of BMI matrix inequalities (6.11), (6.12) and LMI matrix inequalities (6.31) and (6.32) is illustrated on several examples. Note to obtain inequalities (6.31) and (6.32) to the LMI form the standard linearization approach need to be used [22]. In each example, for simulation we have used the nominal plant model. The results obtained using parameter dependent quadratic stability are compared with the results for different quadratic stability variants (*Remark 6.2*). The following 4 variants of the Lyapunov function are used in the design procedure to study the differences between the qualities of the designed controllers in these examples.

- *DP1*: Quadratic stability with respect to uncertain model parameter variation. For this case, the Lyapunov matrix is dependent only on θ and it is in the form

$$P(\theta) = P_0 + \sum_{i=1}^{s} P_i \theta_i \tag{6.33}$$

- *DP2*: Parameter dependent quadratic stability. The Lyapunov matrix depends on both ξ and θ and is given as

$$P(\xi, \theta) = P_0(\xi) + \sum_{i=1}^{s} P_i(\xi)\theta_i \tag{6.34}$$

  where

$$P_j(\xi) = \sum_{i=1}^{N} P_{ji}\xi_i \quad j = 0, 1, 2, \ldots s, \quad \sum_{i=1}^{N} \xi_i = 1$$

- *DP3*: Quadratic stability with respect to gain-scheduled parameters. For this case the Lyapunov matrix is dependent only on ξ

$$P(\xi) = \sum_{i=1}^{N} P_i \xi_i \tag{6.35}$$

- *DP4*: Quadratic stability with respect to both gain-scheduled and uncertain parameters. The Lyapunov matrix is P_0, independent of ξ and θ.

The data for the first example is generated by a computer, the second example is borrowed from [23], where its parameters are slightly modified.

Example 6.1. BMI solution. This example illustrates the design of a robust decentralized gain-scheduled *PI* controller. Parameters of gain-scheduled system (6.1) are as follows: number of scheduling parameters $s = 1$, number of uncertainty domain vertices $N = 2$, system order $n = 5$, input number $m = 2$, output number $l = 2$. The augmented system matrices for *PI* controller [18] are:

$$A_{01} = \begin{bmatrix} -1 & 0.4 & 0.5 & 0 & 0 \\ 0.5 & -2 & 0.7 & 0 & 0 \\ 1 & 1 & -2.5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \; B_{01} = \begin{bmatrix} 1 & 0.2 \\ 0.1 & 0.3 \\ 0.1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{02} = \begin{bmatrix} -1.5 & 0.44 & 0.55 & 0 & 0 \\ 0.52 & -2.2 & 0.97 & 0 & 0 \\ 1.4 & 0.91 & -2.75 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \; B_{02} = \begin{bmatrix} 1.4 & 0.2 \\ 0.11 & 0.23 \\ 0.12 & 2.2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} -0.19 & 0.14 & 0.15 & 0 & 0 \\ 0.15 & -0.2 & 0.17 & 0 & 0 \\ 0.11 & 0.14 & -0.285 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; B_{11} = \begin{bmatrix} 0.14 & 0.05 \\ 0.021 & 0.03 \\ 0.02 & 0.2 \\ 00 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} -0.29 & 0.1 & 0.175 & 0 & 0 \\ 0.175 & -0.12 & 0.19 & 0 & 0 \\ 0.15 & 0.18 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.1 & 0.03 \\ 0.01 & 0.023 \\ 0.015 & 0.19 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For parameters $R = rI_r$, $r = 1$, $Q = qI_q$, $q = 10^{-4}$, $r_0 = 100$ ($P \leq r_0 I$), $\theta_1 \in \langle -1, 1 \rangle$ on the base of Theorem 6.1 we have obtained the following PI robust decentralized controller

DP1:

$$F = \begin{bmatrix} \frac{-0.989s-0.722}{s} \\ \frac{-1.234s-1.257}{s} \end{bmatrix} + \begin{bmatrix} \frac{-0.527s+0.002}{s} \\ \frac{4.51s+0.611}{s} \end{bmatrix} 10^{-4}\theta_1$$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.330$

DP2:

$$F = \begin{bmatrix} \frac{-6.7115s-0.1908}{s} \\ \frac{-0.2214s-1.2565}{s} \end{bmatrix} + \begin{bmatrix} \frac{6.7701s-0.0706}{s} \\ \frac{-0.3869s+0.0024}{s} \end{bmatrix} \theta_1$$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.024$

DP3:

$$F = \begin{bmatrix} \frac{-2.037s-0.373}{s} \\ \frac{-1.794s-0.677}{s} \end{bmatrix} + \begin{bmatrix} \frac{0.6981s+0.2925}{s} \\ \frac{0.6093s+0.1641}{s} \end{bmatrix} 10^{-4}\theta_1$$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.115$

DP4:

$$F = \begin{bmatrix} \frac{0.341s-0.02}{s} \\ \frac{-0.578s-1.702}{s} \end{bmatrix} + \begin{bmatrix} \frac{-0.012s+0.0001}{s} \\ \frac{-0.141s+0.0998}{s} \end{bmatrix} 10^{-3}\theta_1$$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.020$.

To compare the obtained different dynamic properties of the closed-loop system for respective variants of parameter dependent and quadratic stability (*DP1 – DP4*), the following experiments are proposed. Let $\theta_1 = A\sin\omega t$, $(A = 1)$, then the outputs of the closed-loop system oscillate with amplitude A_i, $i = 1, 2, \ldots, l$. We have proposed a dynamic property coefficient (DPC) of the closed-loop system defined by $\lambda_\theta = \frac{\sum_{i=1}^{l} A_i/l}{A}$. Using this DPC coefficient, it can be said that the closed-loop system with smaller DPC better attenuates parameter changes. For demonstration of the proposed experiment, *Fig. 6.1* shows the DPC versus ω of the closed-loop system for all cases of DPi, $i = 1, 2, 3, 4$. For simulation we have used the nominal plant model, e.g. nominal system model is calculated as $A_0 = (A_{01} + A_{02})/2$. Analogically to DPC coefficient above, the

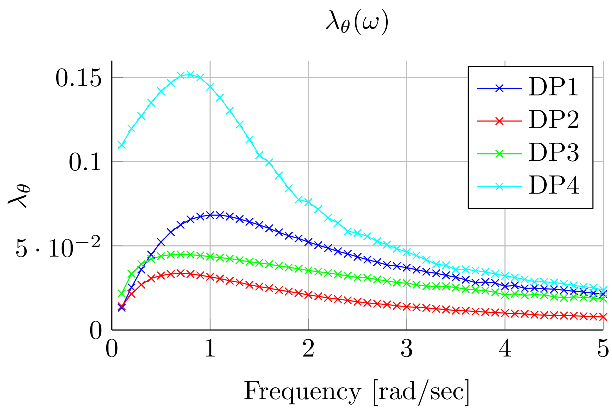

FIGURE 6.1: $\lambda_\theta(\omega)$ at $\theta_1 = \overline{\theta}_1 \sin\omega t$

second dynamic coefficient λ_ξ is defined to assess the influence of uncertainty parameter changes upon the attenuation of the closed-loop system. In the second experiment, the first element $(1, 1)$ of the system matrix has been changed as $A(1, 1) = -1.25 + 0.25\sin\omega t$. The second dynamic property coefficient λ_ξ is given in *Fig. 6.2*

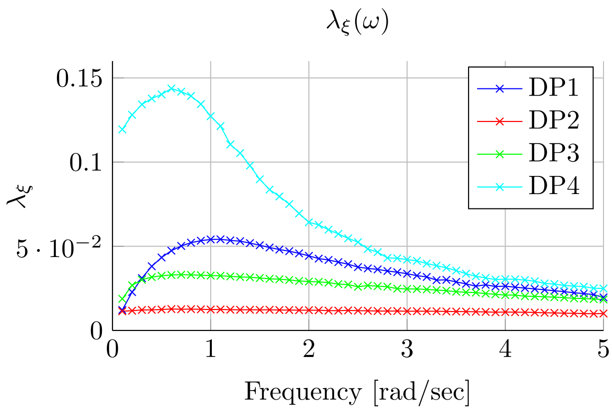

FIGURE 6.2: $\lambda_\xi(\omega)$ at $\theta_1 = 0$, $A_{no}(1, 1) = -1.25 + 0.25\sin\omega t$

Example 6.2. BMI solution. For robust controller design the modified parameters of [23] are as follows $s = 1$, $N = 2$, order of system $n = 4$, $m = 1$, $l = 1$. System matrices for PI controller design are:

$$A_{01} = \begin{bmatrix} -4 & 3 & 5 & 0 \\ 0 & 7 & -5 & 0 \\ 0.1 & -2 & -3 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, A_{02} = \begin{bmatrix} -4.4 & 3.3 & 5.5 & 0 \\ 0 & 7.7 & -5.5 & 0 \\ 0.1 & -1.8 & -3.3 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & -5 & 0 \\ 2 & 5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.8 & 0 & 0.8 & 0 \\ 1.8 & 0 & -4.6 & 0 \\ 1.8 & 4.5 & 1.75 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{01}^T = \begin{bmatrix} 0 & 16 & -10 & 0 \end{bmatrix}, \quad B_{02}^T = \begin{bmatrix} 0 & 13.4 & -12 & 0 \end{bmatrix}$$

$$B_{11}^T = \begin{bmatrix} 1 & -5 & 3.5 & 0 \end{bmatrix}, \quad B_{12}^T = \begin{bmatrix} 0.8 & -4.5 & 3.15 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For parameters $r = 1$, $q = 10^{-6}$, $\theta \in \langle -1, 1 \rangle$, $ro = 10^6$ the following PI controllers have been obtained

DP1:

$F = -3.7699 - 1.9871/s + (-0.1902 - 0.1375/s) \times 10^{-7}\theta_1$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.542$

DP2:

$F = -2.0863 - 0.6452/s + (-0.9418 - 0.0267/s) \times 10^{-7}\theta_1$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.327$

DP3: There are no results. PENBMI failed.

DP4:

$F = -2.6039 - 1.4741/s + (-0.2105 - 0.1827/s) \times 10^{-10}\theta_1$

The closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.589$

The DPC_1 versus ω given in *Fig. 6.3* and *Fig. 6.4* show the dynamic behaviour of the closed-loop system for the case of $DP1$.

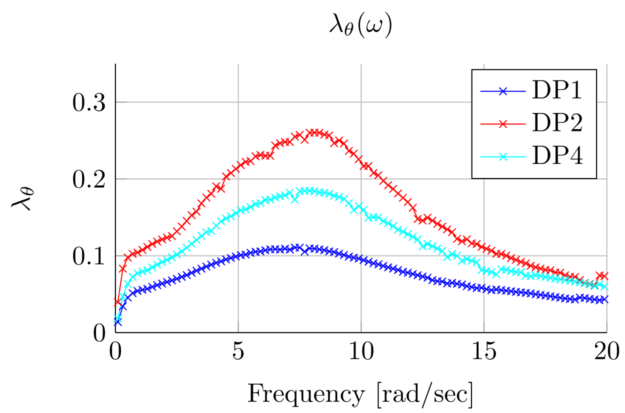

FIGURE 6.3: $\lambda_\theta(\omega)$ at $\theta_1 = \overline{\theta}_1 \sin \omega t$

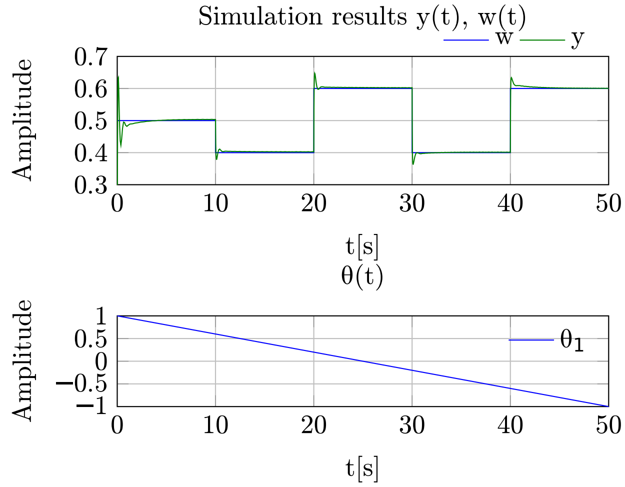

FIGURE 6.4: Dynamic behaviour of the closed-loop system for case $DP1$

Example 6.3. BMI solution. Parameters of gain-scheduled system (6.1) are as follows; $s = 2$, $N = 2$, $n = 3$, $m = 1$, $l = 1$. System matrices for PI controller design are:

$$A_{01} = \begin{bmatrix} -0.0725 & 4.1667 & 0 \\ 0.001 & -0.0735 & 0 \\ 0 & 0.001 & 0 \end{bmatrix}, \; A_{02} = \begin{bmatrix} -0.0725 & 3.8 & 0 \\ 0.0007 & -0.0737 & 0 \\ 0 & 0.001 & 0 \end{bmatrix} \times 10^3$$

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_{21} = \begin{bmatrix} 0 & 833.333 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 0 & 800 & 0 \\ 0 & -1.7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_{01}^T = \begin{bmatrix} 8708.3 & 0 & 0 \end{bmatrix}, \; B_{02}^T = \begin{bmatrix} 8500 & 0 & 0 \end{bmatrix}$$

$$B_{11}^T = \begin{bmatrix} -805 & 0 & 0 \end{bmatrix}, \; B_{12}^T = \begin{bmatrix} -835 & 0 & 0 \end{bmatrix}, \; B_{21}^T = \begin{bmatrix} 2766.7 & 0 & 0 \end{bmatrix}$$

$$B_{22}^T = \begin{bmatrix} 2500 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For parameters $r = 1$, $q = 10^{-4}$. $ro = 10^{10}$, $\theta_1, \theta_2 \in \langle -1, 1 \rangle$ the following PI controllers have been obtained

DP1:

$$F = -0.4108 - 0.0028/s + (0.2914 - 0.0024/s) \times 10^{-12}\theta_1 \\ + (-0.9079 + 0.0028/s) \times 10^{-13}\theta_2$$

Closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.003$

DP2:

$$F = -0.4095 - 0.0035/s + (0.3865 - 0.0051/s) \times 10^{-12}\theta_1 \\ + (-0.1195 + 0.0016/s) \times 10^{-12}\theta_2$$

Closed-loop maximal eigenvalue for $\theta_1 = 0$ is $\lambda_m = -0.004$
DP3: There are no results. PENBMI failed.
DP4: There are no results. PENBMI failed.

The DPC_1 versus ω is given in *Fig.* 6.5 and dynamic behaviour of closed-loop system for $DP2$ is given in *Fig.* 6.6

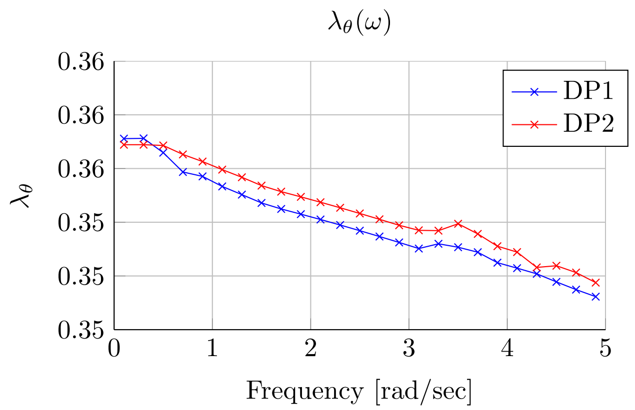

FIGURE 6.5: $\lambda_\theta(\omega)$ at $\theta_1 = \overline{\theta}_1 \sin \omega t$, $\theta_2 = \overline{\theta}_2 \sin(\omega t + 90)$

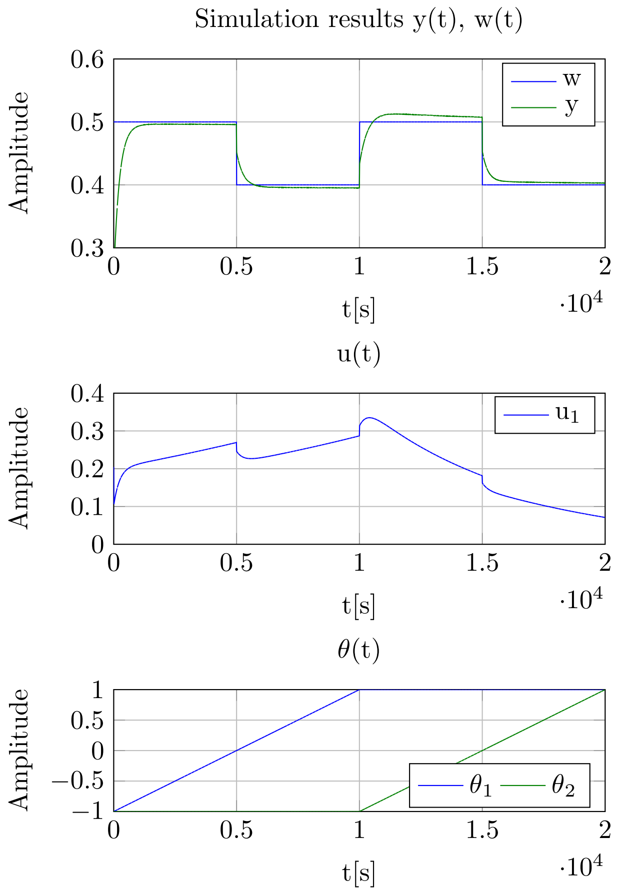

FIGURE 6.6: Dynamic behaviour of closed-loop system for case $DP2$

Example 6.4. LMI solution. The augmented gain-scheduled plant model for an LMI PI robust gain-scheduled controller design are given in Example 6.1. For parameters

$r = 1$, $q = 10^{-5}$, $r_o = 30$, $\theta_m \in \langle 10^{-5}, 1\rangle$ on the base of the *Gain-scheduled robust LMI design procedure* (GSLMIDP) proposed in this paper we have obtained a feasible solution for parameter dependent quadratic stability (the number of Lyapunov functions is $N(s+1)$) the following robust PI controller

$$F = \begin{bmatrix} \frac{-0.1148s-1.5886}{s} \\ \frac{0.257s-1.0515}{s} \end{bmatrix} + \begin{bmatrix} \frac{(-0.0921s+0.1068)}{s} \\ \frac{(-0.0918s+0.0908)}{s} \end{bmatrix} \theta_1$$

The closed-loop maximal eigenvalue for a polytopic system, when $\theta_1 = 0$, is $\lambda_m = -0.1825$. The above results have been obtained in the first step of GSLMIDP after 50 iterations. Dynamic properties of the proposed robust gain-scheduled controller are given in *Fig. 6.7*.

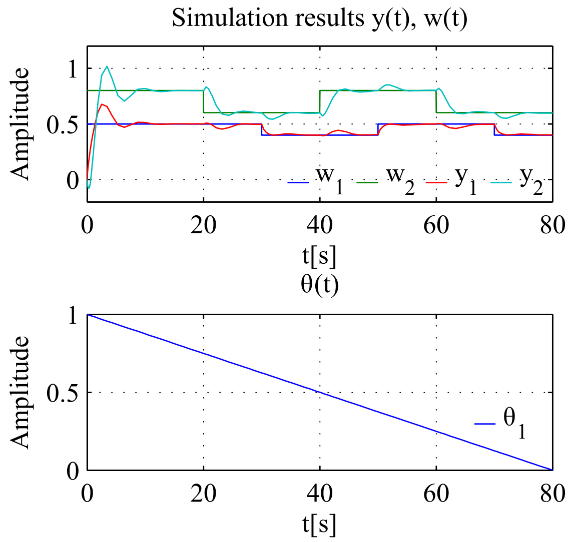

FIGURE 6.7: Dynamic behaviour of closed-loop system

6.6 Conclusion

A novel robust gain-scheduled PI controller design approach has been proposed based on a robust gain-scheduled controller design for an uncertain gain-scheduled polytopic plant model or robust controller design for a uncertain polytopic systems with time varying uncertainty. The obtained results, illustrated on examples, show the applicability of the designed robust gain-scheduled controller and its ability to cope with polytopic model uncertainties. Several forms of parameter dependent/quadratic Lyapunov functions are presented and tested by simulations. Though the proposed robust controller design approach with parameter dependent Lyapunov function does not consider quick changes of parameters (either uncertainty or gain scheduling), simulation results prove the potential ability of the designed closed-loop to withstand also these changes. The obtained results are in the form of BMI and LMI approaches. The proposed approach contributes to the design tools for robust gain-scheduled controllers.

6.7 Appendix

6.7.1 Linearization of (6.32)

$$\overline{M}_{ij} = M_{ijk} + M_{jik} =$$

$$\begin{bmatrix} 0 & N_1^T(A_{ijkc} + A_{jikc}) \\ * & -N_2^T(A_{ijkc} + A_{jikc}) - (A_{ijkc} + A_{jikc})^T N_2 \end{bmatrix}$$

Due to the structure of $\overline{M}_{ij}$, $\overline{M}_{ij} \geq 0$ hold if

$$G_o = -N_2^T(B_{ik}F_j + B_{jk}F_i)C - C^T(B_{ik}F_j + B_{jk}F_i)^T N_2 \geq 0$$

Rewrite above inequality to the following form

$$G_o = \begin{bmatrix} N2^T N2 + (B_{ik}F_jC + B_{jk}F_iC)^T(B_{ik}F_jC + B_{jk}F_iC) & * \\ N_2 + B_{ik}F_jC + B_{jk}F_iC & I \end{bmatrix}$$

Term $G_o(1,1)$ could be linearized known way.

6.7.2 Linearization of (6.31)

Rewrite the first inequality of (6.31) as follows

$$A_c^T P + PA_c + Q + C^T F^T RFC < 0$$

where $A_c = A + BFC$.
Let $G = FC + R^{-1}B^T P$. From $G^T RG$ one obtain

$$C^T F^T B^T P + PBFC = G^T RG - C^T F^T RFC - PBR^{-1}B^T P$$

Substituting above results to first inequality

$$A^T P + PA + Q - PBR^{-1}B^T P < 0$$

Linearization of nonlinear terms $PBR^{-1}B^T P$ gives

$$lin(-PBR^{-1}B^T P) = -PBR^{-1}B^T U - UBR^{-1}B^T P + UBR^{-1}B^T U$$

In LMI iterative procedure for each step one put $U = P$. The same linearization approach can be used to the second inequality of (6.31).

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7

Robust Gain-scheduled PID Controller Design for uncertain LPV systems (Paper 4)

Abstract

A novel methodology is proposed for robust gain-scheduled PID controller design for uncertain LPV systems. The proposed design procedure is based on the parameter-dependent quadratic stability approach. A new uncertain LPV system model has been introduced in this paper. To access the performance quality the approach of a parameter varying guaranteed cost is used which allowed to reach for different working points desired performance. Numerical examples show the benefit of the proposed method.

Keywords: LPV systems, Gain-scheduled controller, Robust controller, Parameter-dependent Lyapunov function, Quadratic gain-scheduled cost function, PID controller.

7.1 Introduction

In real applications a controller must accommodate a plant with changing dynamics. Therefore, controllers based on these models have to be robust in the presence of plant model uncertainty. A practical approach involves scheduling in a family of local controllers in response to the changing plant dynamics [1]. A proposed family of local controllers is implemented using the gain scheduling approach. The above mentioned gain-scheduled designs are guided by two heuristic rules [2]:

- the scheduling variable should vary slowly, and
- the scheduling variable should capture the plants nonlinearities.

In such cases, the designed gain-scheduled controller should be able to stabilize and guarantee a reasonable performance for all operating conditions. The question remains what happens with a closed-loop system if the developed physical nonlinear model or the model obtained through practical identification is not enough precise? In such a case, frequent in applications, there is a need for robust controller to cope with plant model uncertainty.

Various robust controller design methods for gain-scheduled uncertain plant are available in literature. Robust gain-scheduled controllers design to LPV system can be found in [1], where the authors addressed the problem of interpolating in a set of LTI controllers in order to form a gain-scheduled controller with optimal H_∞ performance. The set of admissible interpolated controllers are framed in terms of the robust controller interpolation criteria. For a special uncertain dynamical system, the robust state feedback stabilization problem in the gain scheduling can be found in [3]. In this paper it is shown that a possible advantage of the online measurement of the scheduling parameters is that this always allows linear compensators, whose implementation can be easier than that of nonlinear ones. Design of robust gain-scheduled PI controllers for nonlinear SISO process can be found in [2]. The model uncertainty is assumed to be the difference between the nonlinear model and the linear one. In the paper [4] an input-output approach to the gain-scheduled design of nonlinear controllers is presented. A controller formulation inspired by the Youla-Kucera parametrization to propose a controller structure and design approach that allow the gain scheduling of linear control designs such that a robustly stable nonlinear closed-loop control system is achieved. A robust PID controller is designed in [5].The main feature of the proposed method is that the stability, robustness margin and some performance specification are guaranteed by linear constraints in the Nyquist diagram. The condensing boiler is described by the first order model with a time delay in [6], the problem of attenuation of sinusoidal disturbances with uncertain and arbitrarily time-varying frequencies is solved by synthesis of LPV controller using the L_2 gain method. In [7] the quadratic stability approach is used to design the gain-scheduled controller for each vertex of a plant uncertainty box and the closed-loop system stability is verified by LMI. Other alternative approaches to gain-scheduled controller design can be found in [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. A survey of the gain-scheduled controller design is given in excellent papers [18] and [19].

The above short survey implies that in the references there is no systematic procedure for designing a robust PID gain-scheduled controller. This observation motivated us to solve the following research problem: design a PID robust gain-scheduled controller which should guarantee

- stability and robustness properties of a closed-loop system for all scheduled parameters $\theta \in \Omega_s$ and their rate $\dot{\theta}_i \in \Omega_t$, when the uncertain plant parameters π lie in the given polytopic uncertainty box Ω, that is $\pi \in \Omega$, $\theta \in \Omega_s$, $\dot{\theta} \in \Omega_t$,
- for the closed-loop system ensure for all $\pi \in \Omega$, $\theta \in \Omega_s$ and $\dot{\theta} \in \Omega_t$ guaranteed gain-scheduled performance and parameter dependent quadratic stability.

In this paper the new PID robust gain-scheduled controller design procedure is given.

The paper is organized as follows. *Section 7.2* includes problem formulation of robust PID gain-scheduled controllers design for the original plant uncertainty model and new performance criterion. In *Section 7.3*, sufficient robust stability LMI conditions for the structured gain-scheduled controller are given. Respective conditions for robust controller synthesis are in BMI form. In *Section 7.4*, the results are illustrated on examples to design a PID robust gain-scheduled controller. The final *Section 7.5* brings a conclusion on the obtained results and possible directions in the gain-scheduled controller design field.

Hereafter, the following notational convention will be adopted. Given a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, the inequality $P > 0$ ($P \geq 0$) denotes the positive definiteness (semidefiniteness) matrix. Symbol $*$ denotes a block that is transposed and complex conjugated to the respective symmetrically placed one. Matrices, if not explicitly stated, are assumed to have compatible dimensions. I denotes the identity matrix of corresponding dimensions.

7.2 Problem formulation and preliminaries

Consider a continuous-time linear parameter varying (LPV) uncertain system in the form

$$\begin{aligned} \dot{x} &= \overline{A}(\xi,\theta)x + \overline{B}(\xi,\theta)u \\ y &= Cx \\ \dot{y}_d &= C_d\dot{x} \end{aligned} \tag{7.1}$$

where linear parameter varying matrices

$$\begin{aligned} \overline{A}(\xi,\theta) &= A_0(\xi) + \sum_{i=1}^{s} A_i(\xi)\theta_i \ \in \mathbb{R}^{n\times n} \\ \overline{B}(\xi,\theta) &= B_0(\xi) + \sum_{i=1}^{s} B_i(\xi)\theta_i \ \in \mathbb{R}^{n\times m} \end{aligned} \tag{7.2}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ denote the state, control input and controlled output, respectively. Matrices $A_i(\xi)$, $B_i(\xi)$, $i = 0,1,2,\ldots,s$ belong to the convex set: a polytope with N vertices that can be formally defined as

$$\begin{aligned} &\Omega = \left\{ A_i(\xi),\ B_i(\xi) = \sum_{j=1}^{N} (A_{ij}, B_{ij})\,\xi_j \right\}, \\ &i = 0,1,2,\ldots,s, \quad \sum_{j=1}^{N} \xi_j = 1, \quad \xi_j \geq 0 \end{aligned} \tag{7.3}$$

where s is the number of scheduled parameters; ξ_j, $j = 1, 2, \ldots, N$ are constant or possibly time varying but unknown parameters; matrices A_{ij}, B_{ij}, C, C_d are constant matrices of corresponding dimensions, where C_d is the output matrix for D part of the controller. $\theta \in \mathbb{R}^s$ is a vector of known (measurable) constant or possibly time-varying scheduled parameters. Assume that both lower and upper bounds are available. Specifically

1. Each parameter θ_i, $i = 1, 2, \ldots, s$ ranges between known extremal values

$$\theta \in \Omega_s = \left\{\theta \in \mathbb{R}^s : \ \theta_i \in \langle \underline{\theta}_i, \overline{\theta}_i \rangle, \ i = 1, 2, \ldots, s\right\} \tag{7.4}$$

2. The rate of variation $\dot{\theta}_i$ is well defined at all times and satisfies

$$\dot{\theta} \in \Omega_t = \left\{\dot{\theta}_i \in \mathbb{R}^s : \ \dot{\theta}_i \in \langle \underline{\dot{\theta}}_i, \overline{\dot{\theta}}_i \rangle, \ i = 1, 2, \ldots, s\right\} \tag{7.5}$$

Note that system (7.1), (7.2), (7.3) consists of two type of vertices. The first one is due to the gain-scheduled parameters θ with $T = 2^s$ vertices – θ vertices, and the second set of vertices are due to uncertainties of the system – N, ξ vertices. For robust gain-scheduled "I" part controller design the states of system (7.1) need to be extended in such a way that a static output feedback control algorithm can provide proportional (P) and integral (I) parts of the designed controller. For more details see [20]. Assume that system (7.1) allows PI controller design with a static output feedback.

To access the system performance, we consider an original scheduling quadratic cost function

$$J = \int_0^\infty J(t)dt = \int_0^\infty \left(x^T Q(\theta) x + u^T R u + \dot{x}^T S(\theta) \dot{x}\right) dt \tag{7.6}$$

where

$$Q(\theta) = Q_0 + \sum_{i=1}^{s} Q_i \theta_i, \quad S(\theta) = S_0 + \sum_{i=1}^{s} S_i \theta_i$$

The feedback control law is considered in the form

$$u = F(\theta) y + F_d(\theta) \dot{y}_d \tag{7.7}$$

where

$$F(\theta) = F_0 + \sum_{i=1}^{s} F_i \theta_i, \quad F_d(\theta) = F_{d0} + \sum_{i=1}^{s} F_{di} \theta_i$$

Matrices F_i, F_{di}, $i = 0, 1, 2, \ldots, s$ are the static output PI part and the output derivative feedback gain-scheduled controller. The structure of the above matrices can be prescribed.

The respective closed-loop system is then

$$M_d(\xi, \theta) \dot{x} = A_c(\xi, \theta) x \tag{7.8}$$

where

$$M_d(\xi,\theta) = I - \overline{B}(\xi,\theta)F_d(\theta)C_d$$
$$A_c(\xi,\theta) = \overline{A}(\xi,\theta) + \overline{B}(\xi,\theta)F(\theta)C$$

Let as recall some results about an optimal control of time varying systems [21].

Lemma 7.1. *Let there exists a scalar positive definite function $V(x,t)$ such that $\lim_{t\to\infty} V(x,t) = 0$ which satisfies*

$$\min_{u\in\Omega_u} \left\{ \frac{\delta V}{\delta x} A_c(\theta) + \frac{\delta V}{\delta t} + J(t) \right\} = 0 \tag{7.9}$$

From (7.9) obtained control algorithm $u = u^*(x,t)$ ensure the closed-loop stability and on the solution of (7.1) optimal value of cost function as $J^* = J(x_0, t_0) = V(x(0), t_0)$.

Eq. (7.9) is known as Bellman-Lyapunov equation and function $V(x,t)$ which satisfies to (7.9) is Lyapunov function. For a given concrete structure of Lyapunov function the optimal control algorithm may reduces from "*if and only if*" to "*if*" and for switched systems, robust control, gain-scheduled control and so on to guaranteed cost.

Definition 7.1. Consider a stable closed-loop system (7.8). If there exists a control law u (7.7) which satisfies (7.11) and a positive scalar J^* such that the value of closed-loop cost function (7.6) J satisfies $J < J^*$ for all $\theta \in \Omega_s$ and ξ_j, $j = 1, 2 \ldots, N$ satisfying (7.3), then J^* is said to be a guaranteed cost and u is said to be a guaranteed cost control law for system (7.8).

Let us recall some parameter dependent stability results which provide basic further developments.

Definition 7.2. Closed-loop system (7.8) is parameter dependent quadratically stable in the convex domain Ω given by (7.3) for all $\theta \in \Omega_s$ and $\dot{\theta} \in \Omega_t$ if and only if there exists a positive definite parameter dependent Lyapunov function $V(\xi,\theta)$ such that the time derivative of Lyapunov function with respect to (7.8) is

$$\frac{dV(\xi,\theta,t)}{dt} < 0 \tag{7.10}$$

Lemma 7.2. *Consider the closed-loop system (7.8). Control algorithm (7.7) is the guaranteed cost control law if and only if there exists a parameter dependent Lyapunov function $V(\xi,\theta)$ such that the following condition holds [21]*

$$B_e(\xi,\theta) = \min_u \left(\frac{dV(\xi,\theta,t)}{dt} + J(t) \right) \le 0 \tag{7.11}$$

Uncertain closed-loop system (7.8) conforming to *Lemma 7.2* is called robust parameter dependent quadratically stable with guaranteed cost.

We proceed with the notion of multi-convexity of a scalar quadratic function [22].

Lemma 7.3. *Consider a scalar quadratic function of* $\theta \in \mathbb{R}^s$

$$f(\theta) = \alpha_0 + \sum_{i=1}^{s} \alpha_i \theta_i + \sum_{i=1}^{s} \sum_{j>i}^{s} \beta_{ij} \theta_i \theta_j + \sum_{i=1}^{s} \gamma_i \theta_i^2 \tag{7.12}$$

and assume that if $f(\theta)$ *is multiconvex that is*

$$\frac{\partial^2 f}{\partial \theta_i^2} = 2\gamma_i \geq 0, \quad i = 1, 2, \ldots, s$$

Then $f(\theta)$ *is negative in the hyper rectangle (7.4) if and only if it takes negative values at the vertices of (7.4), that is if and only if* $f(\theta) < 0$ *for all vertices of the set given by (7.4). For decrease the conservatism of Lemma 7.3 the approach proposed in [22] can be used.*

In this paragraph for uncertain gain scheduling system (7.1) we have proposed to use a model uncertainty in the form of a convex set with N vertices defined by (7.3). Furthermore, we consider the new type of performance (7.6) to obtain the closed-loop system guaranteed cost.

7.3 Main Results

This section formulates the theoretical approach to robust PID gain-scheduled controller design for polytopic system (7.1), (7.2), (7.3) which ensures closed-loop system parameter dependent quadratic stability and a guaranteed cost for all gain scheduling parameters $\theta \in \Omega_s$, and $\dot{\theta} \in \Omega_t$. The main result on robust stability for the gain-scheduled control system is given in the next theorem.

Theorem 7.1. *The closed-loop system (7.8) is robust parameter dependent quadratically stable with a guaranteed cost if there exist positive definite matrix* $P(\xi, \theta) \in \mathbb{R}^{n \times n}$*, matrices* N_1*,* $N_2 \in \mathbb{R}^{n \times n}$ *positive definite (semidefinite) matrices* $Q(\theta)$*,* R*,* $S(\theta)$ *and gain-scheduled controller (7.7) such that*

a)

$$\begin{aligned} L(\xi, \theta) = W_0(\xi) + \sum_{i=1}^{s} W_i(\xi)\theta_i + \\ + \sum_{i=1}^{s} \sum_{j>i}^{s} W_{ij}(\xi)\theta_i\theta_j + \sum_{i=1}^{s} W_{ii}\theta_i^2 < 0 \end{aligned} \tag{7.13}$$

b)

$$W_{ii}(\xi) \geq 0, \quad \theta \in \Omega_s, \ i = 1, 2, \ldots, s \tag{7.14}$$

where we consider parameter dependent Lyapunov matrix

$$P(\xi,\theta) = P_0(\xi) + \sum_{i=1}^{s} P_i(\xi)\theta_i > 0 \tag{7.15}$$

the above matrices (7.13) *and* (7.14) *are given as follows:*

$$\begin{aligned}
W_0(\xi) &= \begin{bmatrix} W_{110}(\xi) & W_{120}(\xi) \\ * & W_{220}(\xi) \end{bmatrix} \\
W_{110}(\xi) &= S_0 + C_d^T F_{d0}^T R F_{d0} C_d \\
&\quad + N_1^T \left(I - B_0(\xi) F_{d0} C_d\right) \\
&\quad + \left(I - B_0(\xi) F_{d0} C_d\right)^T N_1 \\
W_{120}(\xi) &= -N_1^T \left(A_0(\xi) + B_0(\xi) F_0 C\right) \\
&\quad + \left(I - B_0(\xi) F_{d0} C_d\right)^T N_2 + P_0(\xi) \\
&\quad + C_d^T F_{d0}^T R F_0 C \\
W_{220}(\xi) &= -N_2^T \left(A_0(\xi) + B_0(\xi) F_0 C\right) \\
&\quad - \left(A_0(\xi) + B_0(\xi) F_0 C\right)^T N_2 + Q_0 \\
&\quad + C^T F_0^T R F_0 C + \textstyle\sum_{j=1}^{s} P_j(\xi)\dot{\theta}_i \\
W_i(\xi) &= \begin{bmatrix} W_{11i}(\xi) & W_{12i}(\xi) \\ * & W_{22i}(\xi) \end{bmatrix} \\
W_{11i}(\xi) &= S_i + C_d^T \left(F_{d0}^T R F_{di} + F_{di}^T R F_{d0}\right) C_d \\
&\quad - N_1^T \left(B_0(\xi) F_{di} + B_i(\xi) F_{d0}\right) C_d \\
&\quad - \left[\left(B_0(\xi) F_{di} + B_i(\xi) F_{d0}\right) C_d\right]^T N_1 \\
W_{12i}(\xi) &= -N_1^T \left(A_i(\xi) + B_0(\xi) F_i + B_i(\xi) F_0\right) C \\
&\quad - \left(B_i(\xi) F_{d0} C_d\right)^T N_2 + P_i(\xi) \\
&\quad + C_d^T \left(F_{di}^T R F_0 + F_{d0}^T R F_i\right) C \\
W_{22i}(\xi) &= -N_2^T \left(A_i(\xi) + \left(B_0(\xi) F_i + B_i(\xi) F_0\right) C\right) \\
&\quad - \left[A_i(\xi) + \left(B_0(\xi) F_i + B_i(\xi) F_0\right) C\right]^T N_2 \\
&\quad + Q_i + C^T \left(F_0^T R F_i + F_i^T R F_0\right) C \\
W_{ij}(\xi) &= \begin{bmatrix} W_{11ij}(\xi) & W_{12ij}(\xi) \\ * & W_{22ij}(\xi) \end{bmatrix} \\
W_{11ij}(\xi) &= C_d^T \left(F_{di}^T R F_{dj} + F_{dj}^T R F_{di}\right) C_d \\
&\quad - N_1^T \left(B_i(\xi) F_{dj} + B_j(\xi) F_{di}\right) C_d \\
&\quad - C_d^T \left(B_i(\xi) F_{dj} + B_j(\xi) F_{di}\right)^T N_1 \\
W_{12ij}(\xi) &= -N_1^T \left(B_i(\xi) F_j + B_j(\xi) F_i\right) C \\
&\quad - C_d^T \left(B_i(\xi) F_{dj} + B_j(\xi) F_{di}\right)^T N_2 \\
&\quad + C_d^T \left(F_{di}^T R F_j + F_{dj}^T R F_i\right) C \\
W_{22ij}(\xi) &= -N_2^T \left(B_i(\xi) F_j + B_j(\xi) F_i\right) C \\
&\quad - C^T \left(B_i(\xi) F_j + B_j(\xi) F_i\right)^T N_2 \\
&\quad + C^T \left(F_i^T R F_j + F_j^T R F_i\right) C \\
W_{ii}(\xi) &= \begin{bmatrix} W_{11ii}(\xi) & W_{12ii}(\xi) \\ * & W_{22ii}(\xi) \end{bmatrix} \\
W_{11ii}(\xi) &= C_d^T F_{di}^T R F_{di} C_d - N_1^T B_i(\xi) F_{di} C_d \\
&\quad - C_d^T F_{di}^T B_i(\xi)^T N_1
\end{aligned}$$

$$\begin{aligned} W_{12ii}(\xi) &= -N_1^T B_i(\xi) F_i C - C_d^T F_{di}^T B_i^T(\xi) N_2 \\ &+ C_d^T F_{di}^T R F_i C \\ W_{22ii}(\xi) &= -N_2^T B_i(\xi) F_i C - C^T F_i^T B_i^T(\xi) N_2 \\ &+ C^T F_i^T R F_i C \end{aligned}$$

Due to space we provide only the outline of the proof. The proof is based on *Lemma 7.2* and *7.3*. The time derivative of the Lyapunov function $V(\xi, \theta) = x^T P(\xi, \theta) x$ is

$$\frac{dV(\xi,\theta)}{dt} = \left[\dot{x}^T \; x^T\right] \begin{bmatrix} 0 & P(\xi,\theta) \\ P(\xi,\theta) & P(\xi,\dot{\theta}) \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{7.16}$$

where

$$P(\xi, \dot{\theta}) = \sum_{i=1}^{s} P_i(\xi)\dot{\theta}$$

To isolate two matrices (system and Lyapunov) introducing matrices N_1, N_2 in the following way

$$\left[2N_1\dot{x} + 2N_2 x\right]^T \left[M_d(\xi\theta)\dot{x} - A_c(\xi,\theta)\right] = 0 \tag{7.17}$$

and substituting (7.17), (7.16), $J(t)$ (7.6) and control law (7.7) to (7.11), after some manipulation one obtains

$$B_e(\xi,\theta) = \left[\, \dot{x}^T \;\; x^T \,\right] \begin{bmatrix} W_{11}(\xi) & W_{12}(\xi) \\ W_{12}{}^T(\xi) & W_{22}(\xi) \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{7.18}$$

where

$$\begin{aligned} W_{11} &= S(\theta) + C_d^T F_d^T(\theta) R F_d(\theta) C_d + N_1^T M_d(\xi,\theta) \\ &+ M_d^T(\xi,\theta) N_1 \\ W_{12} &= -N_1^T A_c(\xi,\theta) + M_d^T(\xi,\theta) N_2 + P(\xi,\theta) \\ &+ C_d^T F_d^T(\theta) R F(\theta) C \\ W_{22} &= -N_2^T A_c(\xi,\theta) - A_c^T(\xi,\theta) N_2 + Q(\theta) \\ &+ C^T F^T(\theta) R F(\theta) C + P(\xi,\dot{\theta}) \end{aligned}$$

Eq. (7.18) immediately implies (7.13), which proves the sufficient conditions of *Theorem 7.1*.

Eq.'s (7.13) and (7.14) are linear with respect to uncertain parameter ξ_j, $j = 1, 2, \ldots, N$, therefore (7.13) and (7.14) have to hold for all $j = 1, 2, \ldots, N$. For the known gain-scheduled controller parameters, inequalities (7.13) and (7.14) reduce to LMI, for gain-scheduled controller synthesis problem (7.13) (7.14) are BMI.

Remark 7.1. *Theorem 7.1* can be used for a quadratic stability test, where Lyapuunov function matrices (matrix) are either independent of parameter ξ_j, $j = 1, 2, \ldots, N$ or parameter θ_i, $i = 1, 2, \ldots, s$ or both as listed below.

1. Quadratic stability with respect to model parameter variation. For this case one has $P(\theta) = P_0 + \sum_{i=1}^{s} P_i \theta_i$. This Lyapunov function should withstand arbitrarily fast model parameter variation in the convex set (7.3)

2. Quadratic stability with respect to gain-scheduled parameters θ. For this case $P_i \to 0$, $i = 1, 2 \ldots, s$ and Lyapunov matrix is $P(\xi, \theta) = P_0(\xi)$. This Lyapunov function should withstand arbitrarily fast θ parameter variations.

3. Quadratic stability with respect to both ξ and θ parameters. Lyapunov matrix is $P(\xi, \theta) = P_0$ and it should withstands arbitrarily fast model and gain-scheduled parameter variation.

7.4 Examples

In this section the robust PID gain-scheduled controller design procedure described in Section 7.3 is illustrated on three examples. Each example is calculated for three quadratic stability approaches (Remark 7.1) and for parameter dependent quadratic stability, that is

QS1: Quadratic stability with respect to uncertain model parameter variation. For this case the Lyapunov matrix is in the form

$$P(\xi, \theta) = P_0 + \sum_{i=1}^{s} P_i \theta_i \tag{7.19}$$

QS2: Parameter dependent quadratic stability. The Lyapunov matrix is given as

$$P(\xi, \theta) = P_0(\xi) + \sum_{i=1}^{s} P_i(\xi) \theta_i \tag{7.20}$$

where

$$P_j(\xi) = \sum_{v=1}^{N} P_{jv} \xi_v, \quad j = 0, 1, 2, \ldots, s, \quad \sum_{i=1}^{N} \xi_i = 1$$

QS3: Quadratic stability with respect to gain-scheduled parameters. The Lyapunov matrix is in the form

$$P(\xi, \theta) = \sum_{i=1}^{N} P_i \xi_i \tag{7.21}$$

QS4: Quadratic stability with respect to both gain-scheduled and model uncertain parameters. The Lyapunov matrix is

$$P(\xi, \theta) = P_0 \tag{7.22}$$

Example 1. The first numerical example has been borrowed from [4] with a small modification. Consider a simple linear plant with parameter varying coefficients

$$\begin{aligned} \dot{x}(t) &= \gamma a(\alpha) x(t) + \gamma b(\alpha) u(t) \\ y(t) &= x(t) \end{aligned} \tag{7.23}$$

where

$$a(\alpha) = -6 - \frac{2}{\pi}\arctan\left(\frac{\alpha}{20}\right)$$
$$b(\alpha) = \frac{1}{2} + \frac{5}{\pi}\arctan\left(\frac{\alpha}{20}\right)$$

$\gamma \in \langle 0.9,\ 1.1\rangle$ being an unknown but constant coefficient and $\alpha \in \langle 0,\ 100\rangle$ a measurable parameter. Let us take 3 working points $\alpha = 0, 30, 100$ where one obtains two models for $\gamma = 0.9$ and $\gamma = 1.1$ (for each working point). The above models have been recalculated to the form (7.1), (7.2), (7.3). Due to I part controller the extended plant models are

$$A_0 = \begin{bmatrix} -6.4370\gamma & 0 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.3130\gamma & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.1240\gamma & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.5930\gamma \\ 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1.275\gamma \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3110\gamma \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0$$

For parameters $R = rI$, $r = 1$, $Q(\theta) = q_0 I + q_1 I + q_2 I$, $q_0 = 0.1$, $q_1 = q_2 = 0.02$, $S(\theta) = s_0 I + s_1 I + s_2 I$, $s_0 = s_1 = s_2 = 0$, $r_0 = 2000$ $(0 < P(\xi, \theta) < r_0 I)$, $\theta_i \in \langle -1, 1\rangle$, $i = 1, 2$; we have obtained the following PID robust gain-scheduled controller

$$R(s) = R_0(s) + R_1(s)\theta_1 + R_2(s)\theta_2$$

QS1: PENBMI failed
QS2: Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.2498$

$$\begin{aligned} R_0(s) &= -1.3667 - 1.2936/s - 0.07s \\ R_1(s) &= 1.8456 + 0.6652/s + 0.0289s \\ R_2(s) &= 1.431 + 0.5161/s + 0.028s \end{aligned}$$

QS3: Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.0407$

$$\begin{aligned} R_0(s) &= -9.6682 - 0.5575/s + 0.0375s \\ R_1(s) &= 0.5187 + 0.0235/s + 0.0019s \\ R_2(s) &= 1.3277 + 0.0602/s + 0.0049s \end{aligned}$$

QS4: PENBMI failed

The closed-loop dynamic behaviours for *QS3* are given in *Fig. 7.1*, where the black line is the setpoint $w(t)$ and the coloured lines are the measured outputs $y(t)$ at $\alpha = 0, 2, 4, \ldots, 100$. Another closed-loop dynamic behaviours for *QS2*, $\gamma = 1$ are given in *Fig. 7.2*, where $w(t)$ is the setpoint, $y(t)$ is the system output, $u(t)$ is the controller output, θ_1 and θ_2 are calculated scheduled parameters and α is the exogenous signal.

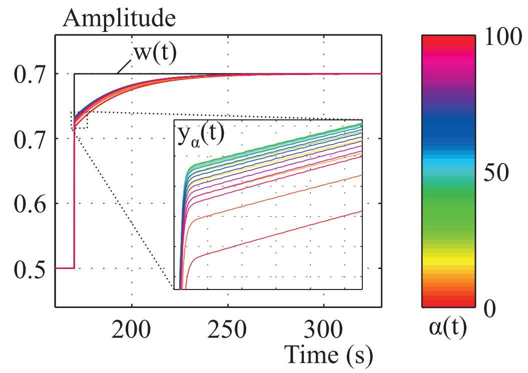

FIGURE 7.1: Simulation results at *QS3*, $\gamma = 1$, $\alpha \in \langle 0,\ 100 \rangle$

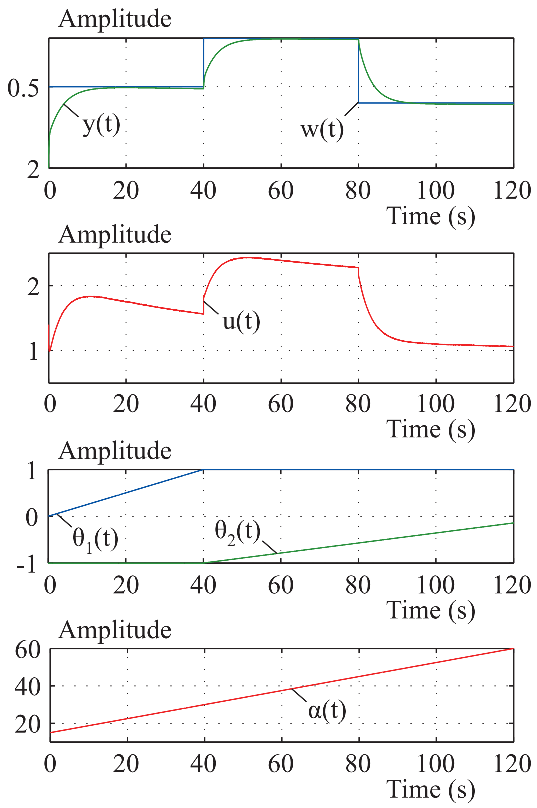

FIGURE 7.2: Simulation results at *QS2*, $\gamma = 1$, $\alpha \in \langle 0,\ 100 \rangle$

Example 2. Second example has been borrowed from [3]. Uncertain model (7.1) is given as follows

$$A(\theta) = \begin{bmatrix} 0.1\gamma & \theta_1 + 4\theta_2 \\ -1 & 0 \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} 0 \\ \gamma\theta_1 + 1.5\theta_2 \end{bmatrix}$$

where $\theta_1 + \theta_2 = 1$, $\theta_i \geq 0$, $i = 1, 2$ and uncertain parameter $\gamma \in \langle 0.9, 1.1 \rangle$. Substituting for $\theta_2 = 1 - \theta_1$ and for $\gamma = 0.9$ or $\gamma = 1.1$ one obtains

$$\begin{aligned} A_{01} &= \begin{bmatrix} 0.09 & 4 \\ -1 & 0 \end{bmatrix}, & A_{02} &= \begin{bmatrix} 0.11 & 4 \\ -1 & 0 \end{bmatrix}, \\ A_{11} &= A_{12} = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}, & B_{01} &= B_{02} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \\ B_{11} &= \begin{bmatrix} 0 \\ -0.6 \end{bmatrix}, & B_{12} &= \begin{bmatrix} 0 \\ -0.4 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 1 \end{bmatrix}, & D &= 0 \end{aligned}$$

For parameters $r = 1$, $\theta_1 \in \langle 0, 1 \rangle$, $q_0 = 0.001$, $s_0 = 0$, $q_1 = 0.0002$, $s_1 = 0$, $r_0 = 20000$ the following PID controller is obtained
QS1: Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.1483$

$$\begin{aligned} R_0(s) &= -4.2735 - 0.7288/s - 0.521s \\ R_1(s) &= -29.0575 - 8.6869/s - 15.9056s \end{aligned}$$

QS2: PENBMI failed
QS3: Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.1129$

$$\begin{aligned} R_0(s) &= 0.0805 - 0.0466/s - 1.9234s \\ R_1(s) &= -0.0649 - 0.4289/s - 15.0506s \end{aligned}$$

QS4: Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.0752$

$$\begin{aligned} R_0(s) &= 0.0304 - 0.0405/s - 1.7515s \\ R_1(s) &= -0.0763 - 0.5508/s - 19.9456 \end{aligned}$$

The closed-loop dynamic behaviours are given in *Fig. 7.3*.

Example 3. Consider the uncertain system (7.1) [9]

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.2\gamma & -0.8 \\ 0.3 & -1.3 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.0 & -0.3\gamma \\ 0.5\gamma & 0 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 0.4 \\ 0.8\gamma \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}\gamma, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

where $\gamma \in \langle 0.9,\ 1.1 \rangle$ constant but uncertain parameter $\theta_1 \in \langle 0, 1 \rangle$. Despite the simplicity the system with state feedback is not quadratically stabilizable with a fixed gain matrix for $\gamma = 1$. Substituting $\gamma = 0.9$ and $\gamma = 1.1$ we obtain the uncertain plant model (7.1). For parameters $r = 1$, $q_0 = 0.0001$, $s_0 = 0$, $q_1 = 0.0001$, $s_1 = 0$ and $r_0 = 20000$ the following robust PID controllers are obtained
QS1: Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.0222$

$$\begin{aligned} R_0(s) &= 1.0149 + 0.0172/s - 1.94333s \\ R_1(s) &= [-0.4446 + 0.6918/s + 11.845s] \times 10^{-14} \doteq 0 \end{aligned}$$

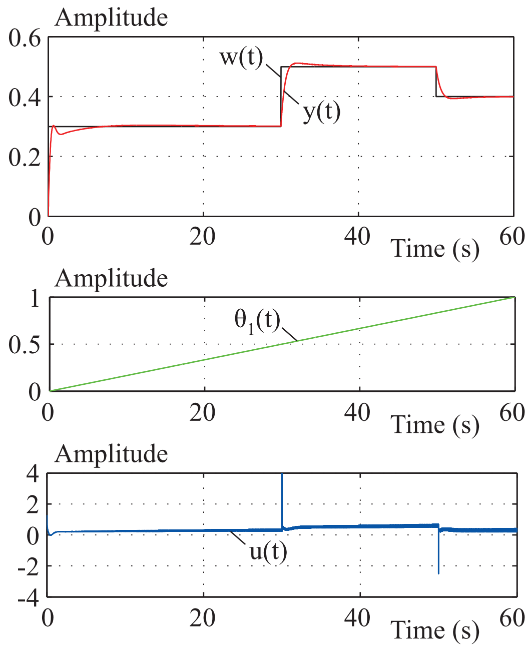

FIGURE 7.3: Simulation results at *QS1*, $\gamma = 1$, $\theta_1 \in \langle 0, 1 \rangle$

QS2, *QS3* and *QS4*: PENBMI failed

When one changes $s_0 = 0.1$, $s_1 = 0.001$ the new PID controller parameters are obtained for *QS1:*

$$R_0(s) = 1.0435 + 0.017/s - 1.7061s$$
$$R_1(s) = [-0.224 + 0.643/s + 10.258] \times 10^{-14} \doteq 0$$

Closed-loop maximal eigenvalue for $\theta_1 = \theta_2 = 0$ is $\lambda_{max} = -0.0209$. The closed-loop dynamic behaviours are given in *Fig. 7.4*.

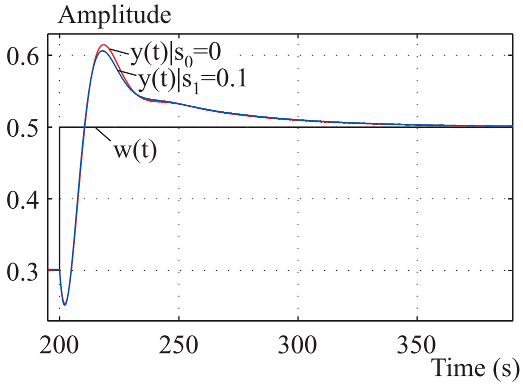

FIGURE 7.4: Simulation results at $\theta_1 = 0$

7.5 Conclusion

A novel design procedure has been proposed for robust gain-scheduled controller design. Several forms of parameter dependent quadratic stability are presented which withstand arbitrarily fast model parameter variation or/and arbitrarily fast gain-scheduled parameter variation. Because of BMI approach the future research should transform BMI to LMI and the obtained design procedure for a polytopic continuous system should be transformed to discrete ones. The proposed approach contributes to the design tools of a robust gain-scheduled controller for uncertain polytopic systems.

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8

Robust Controller Design for T1DM Individualized Model: Gain Scheduling Approach (Paper 5)

Abstract

This paper deals with the robust gain-scheduled controller design for individualized type 1 diabetes mellitus (T1DM) subject model. The controller is designed using LPV model created from T1DM minimal model with two additional subsystems - absorption of digested carbohydrates and subcutaneous insulin absorption. Data collected from continuous glucose monitoring with the help of pharmacodynamics and pharmacokinetics characteristics were used for model identification. The closed-loop stability and cost for all scheduled parameters is guaranteed by the controller design approach. The benefits of the presented approach are shown in the simulation results.

Keywords: LPV system, Robust controller, Gain scheduling, Output feedback, Quadratic stability, Type 1 diabetes mellitus model.

8.1 Introduction

Computer modeling of type 1 diabetes mellitus (T1DM) has attracted considerable attention in the past decade. Patients with T1DM suffer from high levels of glucose concentration due to defective insulin secretion. The lack of insulin is preventing glucose uptake and utilisation by cells. Long-term high glucose concentration results in several health complications. The most common intensified insulin therapy nowadays is based on manual exogenous insulin dosing to either keep the level of basal insulin or to suppress glycemic excursions after a meal. The patient needs to take several fingerstick blood glucose measurements a day and make decisions on insulin doses. A closed-loop blood glucose control would dramatically improve the life of T1DM subjects. Despite the fast development of insulin pumps and continuous glucose measurement systems, a fully autonomous control of glycemia has not been introduced in a commercially available device yet.

The robust control theory is well established for linear systems but almost all real processes are more or less nonlinear. If the plant operating region is small, one can use the robust control approaches to design a linear robust controller where the nonlinearities are treated as model uncertainties. However, for real nonlinear processes, where the operating region is large, the above mentioned controller synthesis is inapplicable. For this reason the controller design for nonlinear systems is nowadays a very determinative and important field of research.

Gain scheduling is one of the most common used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. Many of the early articles were associated with flight control [1] and aerospace [2]. Then, gradually, this approach has been used almost everywhere in control engineering, which was greatly helped with the introduction of LPV systems. Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. They were introduced first by Jeff S. Shamma in 1988 to model gain scheduling. Today the LPV (Linear Parameter-Varying) paradigm has become a standard formalism in systems and controls with lot of researches and articles devoted to analysis, controller design and system identification of these models [3].

The main motivation of our paper were our previous results in gain scheduling [4], [5], [6] and the results from T1DM research [7], [8] and [9]. In this paper a novel robust discrete gain scheduling controller design for Bergman's minimal model of glucose-insulin dynamics coupled with insulin and carbohydrates absorption subsystems is proposed.

Our notations are standard, $D \in \mathbb{R}^{m \times n}$ denotes the set of real $m \times n$ matrices. I_m is an $m \times m$ identity matrix and Z_m denotes a zero matrix. If the size can be determined from the context, we will omit the subscript. $P > 0$ ($P \geq 0$) is a real symmetric, positive definite (semidefinite) matrix.

Organisation of the paper is following. *Section 8.2* includes problem formulation and some preliminaries are given. In *Section 8.3* sufficient stability conditions in the form of BMI and/or LMI are given for the design of a robust discrete gain-scheduled controller. In *Section 8.4* the obtained results are illustrated on the T1DM model.

8.2 Problem formulation and preliminaries

In this section we briefly describe the mathematical model of a T1DM subject, which was based on Bergman's minimal model of insulin-glucose interaction [10]. Later in this work the model will be used as a base for controller design and as a patient simulator for verification of the controller.

Our aim was to adjust the parameters of the proposed model so that the output of the model fits the continuous glucose monitoring (CGM) data of a particular T1DM subject. For identification of specific model parameters we used pharmacokinetics (PK) and pharmacodynamics (PD) measurements (as published in [11, 12]) of the particular insulin prescribed to the patient. The information about ingested carbohydrates was also recorded during data acquisition.

8.2.1 T1DM model

Bergman's minimal model consists of two differential equations in the form

$$\dot{X}(t) = -p_2 X(t) + p_2 S_I \left(I(t) - I_b\right) \tag{8.1a}$$

$$\dot{G}(t) = -\big(S_G + X(t)\big)G(t) + S_G G_b + \left(\frac{1}{V_G}\right) Ra(t) \tag{8.1b}$$

where S_G [1/min] is the rate constant which gives the rate of change of glucose caused by deviation from the basal glucose concentration G_b [mg/dl], parameter S_I [ml/μU/min] is known as the insulin sensitivity index and p_2 [1/min] is a rate constant. Parameter V_G [dl/kg] represents the glucose distribution volume per kilogram of body weight BW [kg]. $G(t)$ [mg/dl] is the blood glucose concentration and signal $X(t)$ [1/min] represents the insulin in remote compartment. Values I_b [μU/ml] and G_b [mg/dl] are the basal insulin concentration and the basal glucose concentration respectively. In a basal steady state we have $X(0) = 0$ and $G(0) = G_b$.

Inputs of the model (8.1) are plasma insulin concentration $I(t)$ [μU/ml] and glucose rate of appearance $Ra(t)$ [mg/kg/min]. Signal $Ra(t)$ can have in general two sources – the absorption of glucose from gastro-intestinal tract (modeled as a subsystem) and direct intravenous glucose administration.

Insulin absorption is modeled as a separate subsystem where the output is insulin concentration $I(t)$ [13, 14]. The subsystem has the form

$$\dot{S}_1(t) = -\left(\frac{1}{T_I}\right) S_1(t) + v(t) \tag{8.2a}$$

$$\dot{S}_2(t) = -\left(\frac{1}{T_I}\right) S_2(t) + \left(\frac{1}{T_I}\right) S_1(t) \tag{8.2b}$$

$$\dot{I}(t) = -k_I I(t) + \left(\frac{1}{T_I}\right)\left(\frac{1}{V_I}\right) S_2(t) \tag{8.2c}$$

where parameter T_I [min] is a time constant of the subsystem, k_I [1/min] is a decay rate of insulin in plasma and parameter V_I [dl/kg] represents a insulin distribution volume per kilogram of body weight. Input $v(t)$ [μU/kg/min] is insulin subcutaneous infusion rate, $S_1(t)$ and $S_2(t)$ [μU/kg] represent the amount of insulin in compartments of the subsystem.

Third subsystem describes the glucose absorption from gastrointestinal tract, i.e. output of the subsystem is the signal $Ra(t)$ [mg/kg/min]. The subsystem is described as follows

$$\dot{D}(t) = -\left(\frac{1}{T_D}\right) D(t) + \left(\frac{1}{T_D}\right) A_G d(t) \tag{8.3a}$$

$$\dot{Ra}(t) = -\left(\frac{1}{T_D}\right) Ra(t) + \left(\frac{1}{T_D}\right) D(t) \tag{8.3b}$$

where parameter T_D [min] is a time constant and A_G [dimensionless] is a friction of ingested carbohydrates which are effectively absorbed. Input $d(t)$ [mg/kg/min] is the rate of carbohydrate ingestion at meal time, i.e. signal $d(t)$ is an impulse with a width of one sampling period while the impulse area corresponds to the amount of ingested carbohydrates.

8.2.2 Identification of model parameters

For identification of model parameters we used data collected from a male T1DM subject aged 14, with $BW = 64.6$ [kg] and using fast-acting insulin NovoRapid (insulin Aspart) from an insulin pump.

8.2.2.1 Insulin absorption subsystem:

The first step in model identification was identifying of insulin absorption subsystem based on pharmacokinetics data of the used insulin. PK data from [12] were used. An average basal insulin infusion rate v_b [μU/kg/min] of the subject during the day is known since data from insulin pump are available. Signal $v(t)$ is a sum of bolus part $v_B(t)$ and basal part v_b.

The aim is to identify the vector of unknown parameters $\Theta_1 = \left[T_I \; k_I \; V_I\right]$ so that the error between simulated insulin concentration $I(t)$ and PK data is minimized. In basal (steady) state for a given v_b we get the basal insulin concentration I_b as the output and $I(t)$ response after a bolus administration. We used the nonlinear least-squares optimization to identify the vector Θ_1.

8.2.2.2 Insulin sensitivity index and insulin action time:

In the next step we identified the parameter related to insulin sensitivity S_I and insulin action time p_2. These parameters determine dynamics of remote insulin signal $X(t)$.

The measuring principle of pharmacodynamics is to maintain glycemia at basal concentration after bolus administration by intravenous glucose infusion. This glucose infusion corresponds to the signal $Ra(t)$ in the equation (8.1b).

If the equation (8.1b) is written in the form

$$\dot{G}(t) = -S_G\left(G(t) - G_b\right) + \frac{1}{V_G}\Big(Ra(t) - V_G X(t) G(t)\Big) \tag{8.4}$$

TABLE 8.1: T1DM identified model parameters

T_I	k_I	V_I	S_I	p_2	S_G	T_D
44.55	0.1645	138.8	0.00159	0.0106	0.032	33.474

TABLE 8.2: Other fixed parameters of the model

V_G	G_b	A_G
1.467	8.5	0.95

and we measure the PK, i.e. $\dot{G}(t) = 0$, then $G(t) \approx G_b \quad \forall t$. It is obvious that parameter S_G has minor influence in order to achieve $\dot{G}(t) = 0$, so we assume $S_G = 0$ during this step of parameter identification.

The aim is to identify vector of unknown parameters $\Theta_2 = \begin{bmatrix} S_I & p_2 \end{bmatrix}$ so that the error between $G(t)$ and G_b is minimized. Signal $Ra(t)$ is given by PD data.

8.2.2.3 Finalizing the model:

At last, remaining parameters S_G and T_D are identified based on CGM data. The data containing both basal and bolus insulin dosing together with the amount of ingested carbohydrates were used as inputs to the model.

Now we are identifying a vector of unknown parameters $\Theta_3 = \begin{bmatrix} S_G & T_D \end{bmatrix}$ so that the error between measured CGM data and the simulator output is minimized. Again, nonlinear least–square optimization was used.

All identified parameters are reported in table 8.1. For the extended description of the identification process, please refer to our preliminary work [9]

8.3 LPV-based robust gain-scheduled controller design

In this section a new LPV model is presented on the base of the nonlinear Bergman's minimal model, which is then used to design a robust discrete LPV-based gain-scheduled controller for T1DM.

8.3.1 LPV model of T1DM

The Bergman's model (1) with the insulin absorbation model (2) can be transfomed to the following LPV model with substitutions $x_1(t) = G(t)$, $x_2(t) = X(t)$, $x_3(t) = S_1(t)$, $x_4(t) = S_2(t)$, $x_5(t) = I(t)$ and $u(t) = v(t)$

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + Bu(t) + W(\theta) \\ y(t) &= Cx(t) \end{aligned} \tag{8.5}$$

where $\theta(t) \in \Omega$ is a vector of scheduled parameters and

$$
\begin{aligned}
A(\theta) &= \begin{bmatrix} -p_1(\theta)+b(\theta) & -a(\theta) & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 & p_3(\theta) \\ 0 & 0 & -\frac{1}{T_i} & 0 & 0 \\ 0 & 0 & \frac{1}{T_i} & -\frac{1}{T_i} & 0 \\ 0 & 0 & 0 & \frac{1}{T_i V_i} & -k_i \end{bmatrix} \\
W(\theta) &= \begin{bmatrix} p_1(\theta)G_b \\ -p_3(\theta)I_b \\ I \\ 0 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \\
C &= \begin{bmatrix} 1 \; 0 \; 0 \; 0 \; 0 \end{bmatrix}
\end{aligned}
$$

furthermore

$$
p_1(\theta) = p_{10} + \sum_{i=1}^{p} p_{1i}\theta_i, \;\; p_3(\theta) = p_{30} + \sum_{i=1}^{p} p_{3i}\theta_i,
$$
$$
a(\theta) = a_0 + \sum_{i=1}^{p} a_i\theta_i, \;\; b(\theta) = b_0 + \sum_{i=1}^{p} b_i\theta_i
$$

The coefficient $a(\theta)$ is used to cover the nonlinear part of (8.1b) $X(t)G(t) \rightarrow x_1x_2$ in the following way

$$
a(\theta) = G(t) \Rightarrow a_0 + a_1\theta_1 = x_1 \Rightarrow a(\theta)x_2 = x_1x_2 \tag{8.6}
$$

where $\theta_1(t) = \frac{y-a_0}{a_1}$. The coefficients a_0 and a_1 were calculated so as to maintain the scheduling parameter θ_1 in the range $\langle -1, 1\rangle$

$$
a_0 = \frac{min(y) + max(y)}{2}, \; a_1 = \frac{min(y) - max(y)}{2} \tag{8.7}
$$

Note, the coefficients a_i, $i = 2, 3, 4, 5$ are equal to zero. Similarly, the coefficient $b(\theta)$ is calculated in the following way

$$
b(\theta)y = R_a(t) \Rightarrow b(\theta) = b_0 + b_2\theta_2 + b_3\theta_3 \tag{8.8}
$$

where coefficients b_0 and b_2 are calculated so as to maintain the scheduling parameter θ_2 in the range $\langle -1, 1\rangle$

$$
b_0 = \frac{min(R_a/y) + max(R_a/y)}{2}
$$
$$
b_2 = \frac{min(R_a/y) - max(R_a/y)}{2}
$$
$$
\theta_2 = \frac{\frac{R_a}{y} - b_0}{b_1}
$$

Furthermore $b_3 = 5$ % of avarage $R_a(t)$ (uncertainty) and $b_i = 0$, $i = 1, 4, 5$ as well as $\theta_3(t) \in \langle -1, 1\rangle$ is unknown but constant parameter describing uncertainty.

For parameters p_1 and p_3 we also considered an uncertainty ($\pm 5\%$)

$$p_1(\theta) = p_{10} + p_{14}\theta_4, \quad p_3(\theta) = p_{30} + p_{35}\theta_5 \tag{8.9}$$

where $p_{10} = p_1$, $p_{14} = 5$ % of p_1, $p_{30} = p_3$, $p_{35} = 5$ % of p_3, $p_{1i} = 0$, $i = 1,2,3,5$, $p_{3i} = 0$, $i = 1,2,3,4$ and θ_4, $\theta_5 \in \langle -1, 1\rangle$ are unknown but constant parameters.

For the robust discrete LPV-based gain scheduling controller design the model (8.5) is transformed to discrete time-space and $W(\theta)$ is neglected, because has no effect on stability.

8.3.2 Robust gain-scheduled controller design

The output feedback gain-scheduled control law is considered for discrete-time PID (often denoted as PSD) controller in the form

$$\begin{aligned} u(k) = K_P(\theta|k)e(k) + K_I(\theta|k)\sum_{i=0}^{k} e(i) \\ + K_D(\theta|k)(e(k) - e(k-1)) \end{aligned} \tag{8.10}$$

where $e(k) = y(k) - w(k)$ is control error, $w(k)$ is reference value and gain matrices $K_P(\cdot)$, $K_I(\cdot)$, $K_D(\cdot)$ are controller parameter matrices[1] (indexes P, I, D means proportional, sum (integral) and first difference (derivative), respectively) in the form

$$\begin{aligned} K_P(\theta|k) &= K_{P0} + \textstyle\sum_{i=1}^{p} K_{Pi}\theta_i(k) \\ K_I(\theta|k) &= K_{I0} + \textstyle\sum_{i=1}^{p} K_{Ii}\theta_i(k) \\ K_D(\theta|k) &= K_{D0} + \textstyle\sum_{i=1}^{p} K_{Di}\theta_i(k) \end{aligned}$$

Note that the number of controller gain matrices is only 2 (for θ_1 and θ_2), the rest 3 (uncertainty) is equal to zero. Because the reference signal $w(k)$ does not influence the closed-loop stability, we assume that it is equal to zero. For $w(k) = 0$, the control law (8.10) can be rewritten as

$$\begin{aligned} u(k) = K_P(\theta|k)y(k) + K_I(\theta|k)\sum_{i=0}^{k} y(i) \\ + K_D(\theta|k)(y(k) - y(k-1)) \end{aligned} \tag{8.11}$$

State space description of PID controllers can be derived in the following way [15]. We can extend the system with two state variables $z(k) = [z_1^T(k),\ z_2^T(k)]^T$ where $z_1(k) = \sum_{i=0}^{k-2} y(i)$ and $z_2(k) = \sum_{i=0}^{k-1} y(i)$, then $y(k-1) = z_2(k) - z_1(k)$. Substituting to (8.11) one obtains

$$\begin{aligned} u(k) = (K_P(\theta|k) + K_I(\theta|k) + K_D(\theta|k))\, y(k) \\ + K_I(\theta|k)z_2(k) - K_D(\theta|k)(z_2(k) - z_1(k)) \end{aligned} \tag{8.12}$$

[1] For SISO systems they are scalars.

Control law (8.12) can be transformed to matrix form

$$u(t) = F(\theta|k)\tilde{y}(k) \tag{8.13}$$

where $\tilde{y} = [y(k),\ z_1(k),\ z_2(k)]^T$ is the extended measurement output vector and

$$F(\theta|k)^T = \begin{bmatrix} K_P(\theta|k) + K_I(\theta|k) + K_D(\theta|k) \\ K_D(\theta|k) \\ K_I(\theta|k) - K_D(\theta|k) \end{bmatrix}$$

Substituting the control law (8.13) to the discrete uncertain LPV system the closed-loop system is obtained in the form

$$\tilde{x}(k+1) = A_c(\theta|k)\,\tilde{x}(k) \tag{8.14}$$

where $\tilde{x}(k) = [x(k), z_1(k), z_2(k)]^T$, $A_c(\theta|k) = A_r(\theta|k) + B_r(\theta|k)F(\theta|k)C_r(\theta|k)$ and

$$A_r(\theta|k) = \begin{bmatrix} A(\theta|k) & 0 & 0 \\ 0 & 0 & I \\ C & 0 & I \end{bmatrix},\ B_r(\theta|k) = \begin{bmatrix} B(\theta|k) \\ 0 \\ 0 \end{bmatrix},$$

$$C_r(\theta|k) = \begin{bmatrix} C & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Remark 8.1. The controller's filter of the derivative (differential) part can be included in the system model.

To access the performance quality a quadratic cost function [16] known from LQ theory is used in this paper, where weighting matrices depends on scheduling parameters [17]. Using this approach we can affect performance quality in each operating point separately. The quadratic cost function is then in the form

$$\begin{aligned} J_{df}(\theta) &= \sum_{k=0}^{\infty}(\tilde{x}(k)^T Q(\theta|k)\,\tilde{x}(k) + u(k)^T R\, u(k) \\ &= \sum_{k=0}^{\infty} J_d(\theta|k) \end{aligned} \tag{8.15}$$

where $Q(\theta|k) = Q_0 + \sum_{i=1}^{p} Q_i\theta_i$, $Q_i = Q_i^T \geq 0$ where $Q_0, Q_i \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite (semidefinite) and definite matrices, respectively. The concept of guaranteed cost control is used in a standard way.

Definition 8.1. Consider the system (8.5) with control algorithm (8.10). If there exists a control law u^* and a positive scalar J^* such that the closed-loop system (8.14) is stable and the value of closed-loop cost function (8.15) satisfies $J \leq J^*$ then J^* is said to be a guaranteed cost and u^* guaranteed cost control law for system (8.5).

Substituting the control law (8.12) to the quadratic cost function (8.15) we can obtain

$$J_d(\theta|k) = \tilde{x}^T \left(Q(\theta|k) + C^T F(\theta|k)^T R\, F(\theta|k)\, C\right) \tilde{x} \tag{8.16}$$

Definition 8.2. [18] The linear closed-loop system (8.14) for $\theta \in \Omega$ is quadratically stable if and only if there exist a symmetric positive definite matrix $P > 0$ and for the first difference of Lyapunov function $V(k) = x^T P x$ along the trajectory of closed-loop system (8.14) holds

$$\Delta V(\theta|k) = A_c(\theta|k)^T P A_c(\theta|k) + P < 0 \tag{8.17}$$

From LQ theory we introduce the well known results.

Lemma 8.1. *[19] Consider the closed-loop system* (8.14)*. Closed-loop system* (8.14) *is affinely quadratically stable with guaranteed cost if and only if the following inequality holds*

$$B_e = \min_u \{\Delta V(\theta|k) + J_d(\theta|k)\} \leq 0 \tag{8.18}$$

for all $\theta(k) \in \Omega$*.*

The main result of this section, the robust discrete gain-scheduled controller design procedure, relies on the concept of multi-convexity, that is, convexity along each direction θ_i of the parameter space. The implications of multiconvexity for scalar quadratic functions are given in the next lemma [20].

Lemma 8.2. *Consider a scalar quadratic function of* $\theta \in \mathbb{R}^p$*.*

$$f(\theta_1, \ldots, \theta_p) = a_0 + \sum_{i=1}^{p} a_i \theta_i + \sum_{i=1}^{p} \sum_{j>i}^{p} b_{ij} \theta_i \theta_j + \sum_{i=1}^{p} c_i \theta_i^2$$

and assume that $f(\theta_1, \ldots, \theta_p)$ *is multi-convex, that is* $\frac{\partial^2 f(\theta)}{\partial \theta_i^2} = 2c_i \geq 0$ *for* $i = 1, 2, \ldots, p$*. Then* $f(\theta)$ *is negative for all* $\theta \in \Omega$ *if and only if it takes negative values at the corners of* θ*.*

Using *Lemma 8.1* and *8.2* the following theorem is obtained

Theorem 8.1. *Closed-loop system* (8.14) *is quadratically stable with guaranteed cost if a positive defined* $P > 0$ *for all* $\theta(k) \in \Omega$ *exists, matrices* $Q_i, R, i = 1, 2, \ldots p$ *and gain-scheduled controller matrices* $F(\theta(k))$ *satisfy*

$$M(\theta(k)) < 0; \quad \theta(k) \in \Omega \tag{8.19}$$

$$M_{ii} \geq 0; \quad i = 1, 2, \ldots p \tag{8.20}$$

where

$$M(\theta) = M_0 + \sum_{i=1}^{p} M_i \theta_i + \sum_{i=1}^{p} \sum_{j>i}^{p} M_{ij} \theta_i \theta_j + \sum_{i=1}^{p} M_{ii} \theta_i^2 \tag{8.21}$$

furthermore

$$\begin{aligned}
M_0 &= \begin{bmatrix} -P+Q_0 & C^T F_0^T & Ac_0^T \\ F_0 C & -R^{-1} & 0 \\ Ac_0 & 0 & -P^{-1} \end{bmatrix} \\
M_i &= \begin{bmatrix} Q_i & C^T F_i^T & Ac_i^T \\ F_i C & 0 & 0 \\ Ac_i & 0 & 0 \end{bmatrix} \\
M_{ij} &= \begin{bmatrix} 0 & 0 & Ac_{ij}^T \\ 0 & 0 & 0 \\ Ac_{ij} & 0 & 0 \end{bmatrix}, \; M_{ii} = \begin{bmatrix} 0 & 0 & Ac_{ii}^T \\ 0 & 0 & 0 \\ Ac_{ii} & 0 & 0 \end{bmatrix} \\
Ac_0 &= A_{r0} + B_{r0} F_0 C_r \\
Ac_i &= A_{ri} + B_{ri} F_0 C_r + B_{r0} F_i C_r \\
Ac_{ij} &= B_{ri} F_j C_r + B_{rj} F i C_r \\
Ac_{ii} &= B_{ri} F_i C_r
\end{aligned}$$

Proof. Proof is based on *Lemma 8.1* and *8.2*. From (8.18) we can obtain

$$\begin{aligned} M(\theta(k)) &= A_c(\theta(k))^T P A_c(\theta(k)) + P \\ &+ Q + C^T F(\theta(k))^T R F(\theta(k)) C < 0 \end{aligned} \tag{8.22}$$

Using Schur complement we obtain

$$M(\theta(k)) = \begin{bmatrix} W_{11} & W_{21}^T & W_{31}^T \\ W_{21} & W_{22} & W_{32}^T \\ W_{31} & W_{32} & W_{33} \end{bmatrix} < 0 \tag{8.23}$$

where

$$\begin{aligned} W_{11} &= -P + Q(\theta(k)) & W_{22} &= -R^{-1} \\ W_{21} &= F(\theta(k))C & W_{32} &= 0 \\ W_{31} &= Ac(\theta(k)) & W_{33} &= -P^{-1} \end{aligned}$$

After we extend (8.23) to affine form we obtain (8.19) and (8.20) which proofs the *Theorem 8.1*. □

Note that *Theorem 8.1* in its presented form is in the form of BMI. One can use a free and open source BMI solver *PENLAB* or we can linearize the nonlinear part of (8.19) to use LMI solver (*LMILAB* or *SEDUMI*).

$$lin(-P^{-1}) \leq X^{-1}(P - X)X^{-1} - X^{-1} \tag{8.24}$$

where in each iteration pores $X = P$. Using this linearization, the element obtaining the nonlinear part (M_0) become as follows

$$M_0 = \begin{bmatrix} -P+Q_0 & C^T F_0^T & Ac_0^T \\ F_0 C & -R^{-1} & 0 \\ Ac_0 & 0 & X^{-1}(P-X)X^{-1} - X^{-1} \end{bmatrix}$$

8.4 Simulation experiments

In this section the proposed robust discrete gain-scheduled controller is verified using the individualized T1DM Bergman's model served as a patient. For controller synthesis the LPV model described in *Section 8.3.1* with parameters presented in *Section 8.2.2*, transformed to discrete time-space with sample time $T_s = 5\,min$ was used. The response of Bergman's model with the discrete LPV model to an insulin bolus is shown in *Fig 8.1*. The disturbance has been considered in the form of mixed meal ingestion. The main objective was to keep the blood glucose concentration levels within normal glycemic range $(3.8 - 10\ mmol/l)$.

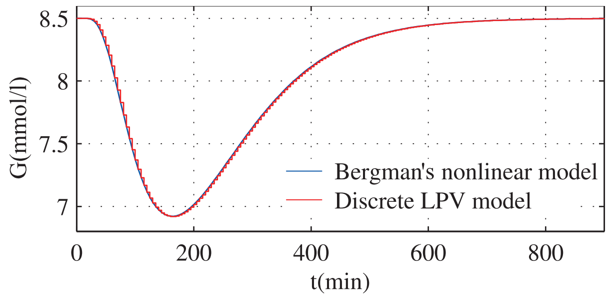

FIGURE 8.1: Bergman's model with the discrete LPV model

The obtained model was extended for robust discrete gain-scheduled PID controller design (8.14). Then using *Theorem 8.1* with weighting matrices $Q = q_i I$, $q_0 = 1 \times 10^{-5}$, $q_1 = 1 \times 10^{-1}$, $q_2 = 1 \times 10^{-2}$, $q_3 = q_4 = q_5 = 0$ $R = rI$, $r = 1$ and $\xi_L \leq P(\theta) \leq \xi_U$, $\xi_U = 1 \times 10^8$, $\xi_L = 1 \times 10^{-5}$, $T_s = 5\,min$ with *LMILAB* one can obtain robust discrete gain-scheduled controller in the form (8.10) where

$$
\begin{aligned}
K_p(\theta) &= -101.1243 - 250.7895\,\theta_1 + 239.9897\,\theta_2 \\
K_i(\theta) &= -5.1342 \times 10^{-8} - 1.2379 \times 10^{-7}\,\theta_1 \\
&\quad -1.0598 \times 10^{-5}\,\theta_2 \\
K_d(\theta) &= -1012.1475 - 5998.7879\,\theta_1 + 0.1235\,\theta_2
\end{aligned}
$$

For the illustration propose simulation experiment results are shown in *Fig. 8.2*. During manual administration of insulin by the T1DM subject, the measured glycemia has been higher than 10 $mmol/l$ during 45% of the monitored time. In the case of automatic dosing controlled by the proposed gain-scheduled algorithm, the time when glycemia reached the level of 10 $mmol/l$ or more was reduced to 9.9% of the simulation time.

8.5 Conclusion

The robust discrete gain-scheduled controller design for Bergman's minimal model of glucose-insulin dynamics coupled with insulin absorption subsystem and carbohydrates absorption subsystem was proposed in this paper. In contrast to publications in literature we presented a completely new LPV description of Bergman's minimal model and a new approach to controller design. The obtained design procedure can be used in

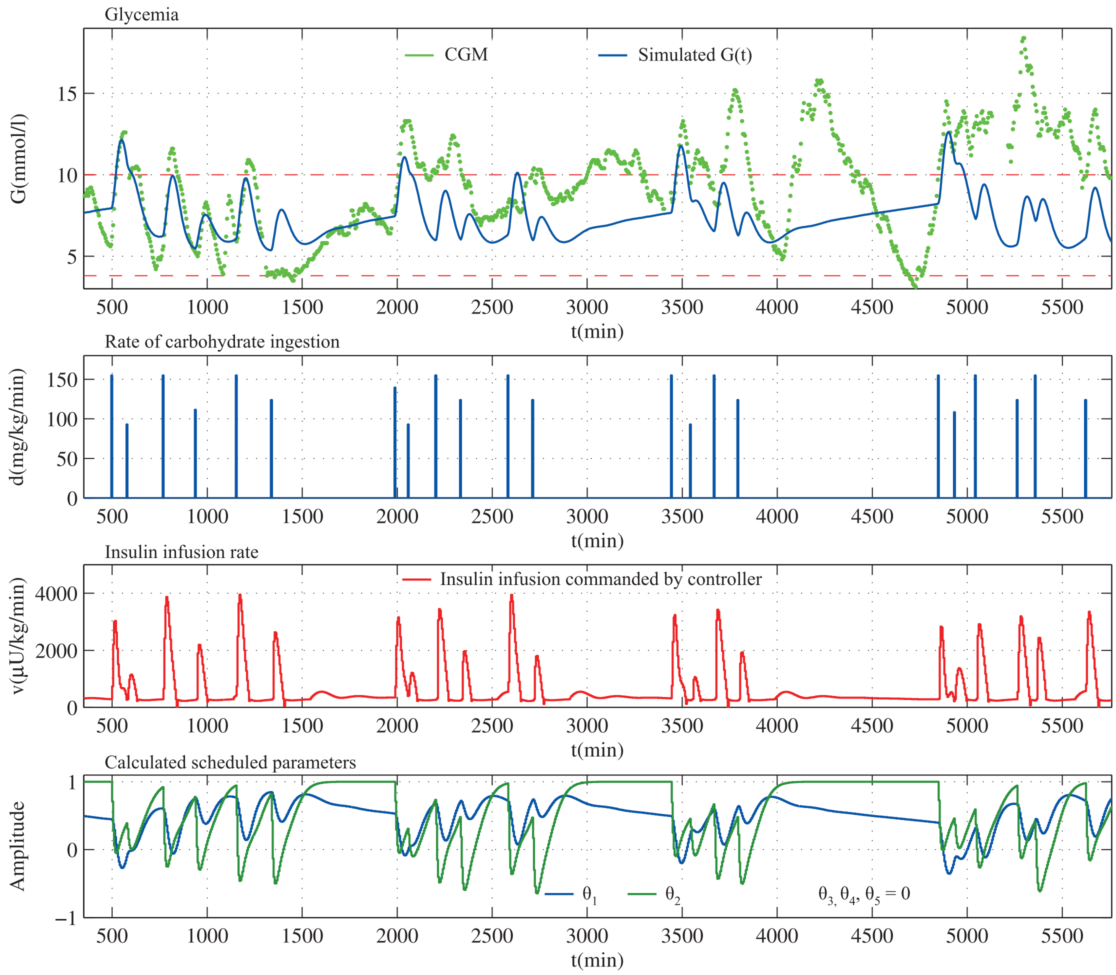

FIGURE 8.2: Simulation results for time period of 4 days

systems where we need to save the operation energy (e.g. low-cost micro-controllers). The presented theory opens new possibilities for further research and study in this area.

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9

Novel approach to switched controller design for linear continuous-time systems (Paper 6)

Abstract

In this paper we study the novel approach to the design of an output feedback switched controller with an arbitrary switching algorithm for continuous-time invariant systems which is described by a novel plant model as a gain-scheduled plant using the multiple quadratic stability and quadratic stability approaches. In the proposed design procedure there is no need to use the notion of the "dwell-time". The obtained results are in the form of bilinear matrix inequalities (BMI). Numerical examples show that in the proposed method the design procedure is less conservative and gives more possibilities than that described in the papers published previously.

Keywords: Switched system, continuous time system, output feedback, quadratic stability, multi quadratic stability.

9.1 Introduction

Switched systems have played an important role in the past decade. Motivation for studying switched systems comes from two facts:

- switched systems have numerous applications in the control of real plants, and
- in real control, there are dynamical systems that cannot be stabilized by any continuous static output/state feedback control law, but a stabilizing switching control scheme can be found.

Switched systems constitute an important class. The switched system consists of a continuous or discrete time system and a switching law that specifies the switching between them. It is pointed out in [1–4] that the stability of switched systems plays an important role in the analysis and design of switched controllers. There are at least two approaches to the stability analysis of switched systems: The quadratic stability approach with a common Lyapunov function gives the stability under an arbitrary switching law, and Multiple Lyapunov functions, which is less conservative. A huge number of references can be found on the switched control of linear discrete-time invariant systems, but in the field of linear continuous-time invariant systems the number of references is rather small. Representatives are the following references [5–9]. In the first paper the authors introduce, into the switched controller design procedure, the notion of dwell-time T_d (minimal time interval between switching). In the stability analysis condition of switched systems the dwell-time is in the term $e^{A_c T_d}$ (A_c-closed-loop matrix (9.4)). The proposed design procedure for stability analysis and switching controller design for the real switching time interval $T \geq T_d$ becomes rather conservative. The above "dwell-time term" complicated the switched controller design procedure for continuous-time systems. In [6] sufficient conditions are given for the stability of linear systems with a dwell-time and with polytopic type parameter uncertainty. A Lyapunov function, in quadratic form for each mode, which is non-increasing at the switching instants is assigned to each mode. During the dwell-time this function varies piecewise linearly in time after switching occurs. The proposed method was applied to stabilization via the state feedback for both nominal and uncertain cases. Since within the dwell-time the Lyapunov function varies piecewise linearly and the real switching time interval $T > T_d$, the switching controller design procedure becomes rather conservative. In [7] the stability analysis problem for a class of switched positive linear systems with average dwell time switching is investigated. Paper [8] investigates the stability of a class of switched linear systems and proposes a number of new results on the stability analysis. A novel analysis method is developed by using the 2-norm technique, and then several stability results are obtained based on the new analysis method. It is shown that the main results obtained in this paper not only guarantee the stability of the systems under arbitrary switching but also provide an algorithm to find the minimum dwell time with which switches make the switched systems stable. Dwell-time switching is a logic for orchestrating the switching between controllers in order to control a proces with a highly uncertain model [9]. The idea of dwell-time switching is to use a parameter tuner which switches controller parameter values rather than continuously adjusting them. Such switched system consisting of two interconnected subsystems, namely a "continuous part" and a "dwell-time switching logic". Together these two subsystems constitute a parameter tuner which is a bona fide hybrid (switched) system. Above system generated "switching logic" to make estimation error (performance) "small" in some sense [9]. Note that implementation of dwell-time switching logic requires capable of minimizing some performance over the set of controller in real time. In this paper the proposed switched controller design procedure for continuous-time systems has none of the above mentioned drawbacks due to dwell-time consideration. An overview of switched systems can be found in [1, 3, 10]. In this paper, new quadratic stability and multi quadratic stability conditions of closed-loop switched systems for arbitrarily switching [3] are given using a new model of switched continuous time linear systems. In the proposed approach to switching controller design for continuous time systems there is no need to use the approach of the "dwell-time" [5, 6]. The proposed switched controller design procedure can easily be expanded to the case of robust switched controller design [11] with multi parameter dependent quadratic stability.

Organization of the paper is as follows. *Section 9.2* includes problem formulation of the switched controller design using the novel proposed model and some preliminary results are given. In *Section 9.3* sufficient stability conditions in the form of BMI for the case of quadratic and multi quadratic approach are given and in *Section 9.4* the obtained results are illustrated on some examples.

Hereafter, the following notational conditions will be adopted. Given a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, the inequality $P > 0$ denotes matrix positive definiteness. Symbol $*$ denotes a block that is transposed and complex conjugated to the respective symmetrically placed one. I denotes the identity matrix of corresponding dimensions.

9.2 Preliminaries and problem formulation

Let us consider a class of linear continuous-time invariant switched systems

$$\begin{aligned} \sum \sigma : \quad \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned} \tag{9.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, $y(t) \in \mathbb{R}^l$ is the output vector of the system to be controlled, and $\sigma \in S = \{1, 2, \ldots, p\}$ is an arbitrarily switching algorithm. The arbitrary switching algorithm σ is a piecewise constant, right continuous function which specifies at each time the index of the active closed-loop system [3, 12], p is the number of switched modes of linear systems and

$$A(\theta) = A_0 + \sum_{i=1}^{p} A_i\theta_i \ , \quad B(\theta) = B_0 + \sum_{i=1}^{p} B_i\theta_i \tag{9.2}$$

where $\sum_{i=1}^{p} \theta_i = 1,\ \theta_i \in \langle 0,\ 1\rangle \in \Omega_\theta,\ i = 1, 2, \ldots, p$ are switching parameters. For calculation of all the above matrices in (9.1) see the example. Note that for switching systems the stable steady state points of switching parameters $\theta_i,\ i = 1, 2, \ldots, p$ are equal to 0 or 1. If the switching parameter $\theta_i,\ i = 1, 2, \ldots, p$ differs from 0 or 1, it is moving to one of the stable points with the rate of θ_i change $\dot{\theta}_i,\ i = 1, 2, \ldots, p$. There are two possibilities for switching parameters:

- the rates of change of switching parameters are infinite, that is $\theta_i = 1$ holds for the i-th mode and $\theta_j = 0$ for $j = 1, 2, \ldots, p,\ j \neq i$ (quadratic stability approach), or

- the rates of change of switching parameters are finite, assume that the system switched from i to j mode. For this case one has $\theta_i + \theta_j = 1$, $\dot{\theta}_i + \dot{\theta}_j = 0$ and $\theta_k = 0$, $k \neq i, j$ (multiple quadratic stability approach).

For both cases θ_i, $i = 1, 2, \ldots, p$ are known and the number of system mode or the number of vertices is equal to "p". Note that number of vertices for polytopic system is 2^p.
The switched output feedback control law is considered in the form

$$u(t) = F(\theta)y(t) = F(\theta)Cx(t) \tag{9.3}$$

where

$$F(\theta) = F_0 + \sum_{i=1}^{p} F_i \theta_i$$

The structure of matrices F_i, $i = 0, 1, \ldots, p$ can be prescribed, for decentralized control structure F_i is a block diagonal matrix and so on. The closed-loop system is obtained from Eqs. (9.1) and (9.3)

$$\dot{x}(t) = \left(A(\theta) + B(\theta)F(\theta)C\right)x(t) = A_c(\theta)x(t) \tag{9.4}$$

To access the system performance, we consider a standard positive definite quadratic cost function with respect to state x and control u.

$$J = \int_0^\infty \left(x(t)^T(t)Qx(t) + u^T(t)Ru(t)\right)dt = \int_0^\infty J(t)dt \tag{9.5}$$

Let us recall some results on the optimal control of time varying systems.

Lemma 9.1. *[13] Let there exists a scalar positive definite function $V(x,t)$ such that $\lim_{t\to\infty} V(x,t) = 0$ which satisfies*

$$\min_{u(t)\in\Omega_u} \left\{ \frac{\partial V}{\partial x} A_c(\theta) + \frac{\partial V}{\partial t} + J(t) \right\} = 0 \tag{9.6}$$

The control algorithm $u(t) = u^*(x,t) \in \Omega_u$ obtained from (9.6) ensures closed-loop stability and the optimal value of the cost function (9.5) as $J^* = J(x_0, t_0) = V(x(0), t_0)$.

Equation (9.6) is known as the Bellman-Lyapunov equation and function $V(x,t)$ which satisfies (9.6) is the Lyapunov function. For a particular structure of the Lyapunov function the optimal control algorithm reduces from "if and only if" to "if", and for switched systems, robust control and so on to a guaranteed cost.

Definition 9.1. Consider system (9.1) and controller (9.3). If there exist a control law $u^*(x,t)$ and a positive scalar J^* such that the respective closed-loop system (9.4) is stable and the value of the closed-loop cost function (9.5) satisfies $J \leq J^*$, then J^* is said to be the guaranteed cost and $u^*(x,t)$ is said to be the guaranteed cost control law for system (9.4).

Definition 9.2. [4] The switched linear closed-loop system (9.4) is said to be quadratically stabilizable via output feedback if there exists a Lyapunov function of the form $V = x(t)^T Px(t),\ P > 0$, a positive number $\epsilon > 0$ and a arbitrary switching rule σ such that

$$\frac{dV}{dt} \leq -\epsilon x(t)^T x(t) \tag{9.7}$$

Definition 9.3. The switched linear closed-loop system (9.4) is said to be multiple quadratically stabilizable via output feedback if there exists a Lyapunov function of the form $V(\theta) = x(t)^T P(\theta)x(t)$, $P(\theta) > 0$, a positive number $\epsilon > 0$ and a arbitrary switching rule σ such that

$$\frac{dV(\theta)}{dt} \leq -\epsilon x(t)^T x(t) \tag{9.8}$$

Lemma 9.2. *[14], [3] Consider a closed-loop system (9.4) with control algorithm (9.3). Control algorithm (9.3) will be a cost guaranteed algorithm if there exists a positive scalar ε such that for the time derivative of the positive definite Lyapunov function the following condition holds*

$$B_e = \min_{u(t)\in\Omega_u} \left\{ \frac{\partial V}{\partial x} A_c(\theta) + \frac{\partial V}{\partial t} + J(t) \right\} \leq -\varepsilon x(t)^T x(t) \tag{9.9}$$

when $\varepsilon \to 0$.

The following lemma plays an important role in the next development [15]

Lemma 9.3. *Consider a scalar quadratic function of θ_i*

$$f(\theta) = a_0 + \sum_{i=1}^{p} a_i\theta_i + \sum_{i=1}^{p}\sum_{j=i}^{p} b_{ij}\theta_i\theta_j \tag{9.10}$$

and assume that $f(\theta)$ is multiconvex, that is

$$\frac{\partial^2 f(\theta)}{\partial\theta_i^2} \geq 0, \quad i = 1, 2, \ldots, p \tag{9.11}$$

then $f(\theta)$ is negative in the hyper rectangle $\theta_i \in \langle 0,\, 1\rangle$, $i = 1, 2, \ldots, p$, if and only if it takes negative values at the corners, that is if and only if $f(\theta) < 0$ for $\theta_i = 0$ or $\theta_i = 1$, $i = 1, 2, \ldots, p$.

9.3 Switched controller design

In this paragraph two methods of switched controller design for linear continuous-time invariant systems are presented. The first method is connected with the notion of quadratic stability with respect to θ. We will assume that the rate of θ change is infinite. Most of the literature on switched controller design concentrates on the case, where switching can occur immediately, thus the rate of change of the switching signal is infinite (ideal switching). In some real cases the rate of change of the switching signal is finite (non ideal switching). This assumption will be used in the second approach to obtain the switched controller design procedure. In the references on the design of switched controllers for continuous-time systems there are no solution for the case of a finite rate of change of switching signal. In this paper we for the first time set up the case of the finite rate of change of the switching signal using the multi quadratic stability approach to the closed loop switched systems. In the proposed approach to the switching controller design for continuous time systems there is no need to use the approach of the "dwell-time" [5, 6], which very complicated the switched controller design procedure.

9.3.1 Quadratic stability approach

The quadratic stability approach to the design of the switched controller for continuous and discrete-time systems is well established. Because of the new model for continuous time systems (9.1) in this part of the paper the proposed method is connected with the notion of quadratic stability with respect to θ. We will assume that the rate of θ change is infinite (ideal switching). From (9.9) the following lemma is obtained.

Lemma 9.4. *Closed-loop $A_c(\theta)$ is quadratically stable with guaranteed cost if there exists a positive definite Lyapunov matrix P such that the following inequality holds*

$$A_c^T(\theta)P + PA_c(\theta) + Q + C^T F(\theta)^T RF(\theta)C \leq 0 \tag{9.12}$$

Substituting $A_c(\theta)$ (9.4) to (9.12) and using *Lemmas 9.2* and *9.3* the following quadratic stability conditions are obtained.

Theorem 9.1. *Closed-loop system (9.4) is quadratically stable with guaranteed cost if there exists a Lyapunov matrix $P > 0$ and matrices $Q \geq 0$, $R > 0$ such that for an arbitrarily switching rule σ the following matrix inequalities hold*

a.)

$$\begin{aligned}(A_0 + B_0F_0C)^T P + P(A_0 + B_0F_0C)\\ +Q + C^T F_0^T RF_0C = M_0 \leq 0\end{aligned} \tag{9.13}$$

b.)

$$M_0 + M_i + M_{ii} \leq 0, \quad i = 1, 2, \ldots, p \tag{9.14}$$

where

$$\begin{aligned}M_i = (A_i + B_0F_iC + B_iF_0C)^T P\\ + P(A_i + B_0F_iC + B_iF_0C)\\ + C^T F_i^T RF_0C + C^T F_0^T RF_iC\end{aligned} \tag{9.15}$$

and

$$M_{ii} = (B_iF_iC)^T P + PB_iF_iC + C^T F_i^T RF_iC \tag{9.16}$$

c.)

$$M_{ii} \geq 0, \quad for\, i = 1, 2, \ldots, p \tag{9.17}$$

provided that $\theta_i = 0$ *or* $\theta_i = 1$

Proof
Proof is based on (9.9), *Lemma 9.2* and *Lemma 9.3* and goes the same line as proof of *Theorem 9.2*.
Equation (9.14) implies that for stability analysis of the switched system we have obtained linear matrix inequalities (LMI) but for the switched controller design we have obtained bilinear matrix inequalities (BMI).

9.3.2 Multiple Lyapunov function approach

In this subsection we will assume that for some realistic cases the switching signal rate of change is finite (non ideal switching). As we mentioned above in the references, there is no solution for these cases. The obtained results for a finite rate of change of switching signal open the new possibilities for the designer (practical realization) and for the theory of the switched controller design procedure, i.e., the design of a switched robust controller, design of a switched controller for some type of nonlinear systems and so on. Specifically

- for switching parameters it holds $\sum_{i=1}^{p} \theta_i = 1$,
  $\theta_i \in \langle 0\ 1 \rangle \in \Omega_\theta$.
- the rate of switching parameter variation $\dot{\theta}_i$ is well defined at all times and satisfies known boundaries

$$\dot{\theta}_i \in \Omega_t = \left\{ \dot{\theta}_i \in \left\langle \underline{\dot{\theta}}_i, \bar{\dot{\theta}}_i \right\rangle, i = 1, 2, \ldots, p \right\} \tag{9.18}$$

$$\sum_{i=1}^{p} \dot{\theta}_i = 0.$$

Main results for the switched controller design using the multiple quadratic stability approach are given in the next theorem.

Theorem 9.2. *Closed-loop system (9.4) is multiple quadratically stable with guaranteed cost if there exist* $p + 1$ *symmetric matrices* $P_0, P_1, \ldots, P_p$ *such that* $P_0 + \sum_{i=1}^{p} P_i \theta_i > 0$ *is positive definite for all switching parameters* $\theta_i \in \Omega_\theta$, $\dot{\theta}_i \in \Omega_t$ *switched controller parameters* $F(\theta)$ *satisfying*

$$\begin{aligned} W(\theta) &\leq 0 \\ W_{ii} \geq 0, &\quad i = 1, 2, \ldots, p \end{aligned} \tag{9.19}$$

for an arbitrary switching rule σ.

where

$$W(\theta) = W_0 + \sum_{i=1}^{p} W_i\theta_i + \sum_{i=1}^{p}\sum_{j=1}^{p} W_{ij}\theta_i\theta_j$$

$$W_0 = \begin{bmatrix} w_{110} & w_{120} \\ w_{120}^T & w_{220} \end{bmatrix}; W_i = \begin{bmatrix} w_{11i} & w_{12i} \\ w_{12i}^T & w_{22i} \end{bmatrix}$$

$$W_{ij} = \begin{bmatrix} w_{11ij} & w_{12ij} \\ w_{12ij}^T & w_{22ij} \end{bmatrix}$$

$$w_{110} = N_1 + N_1^T, \quad w_{11i} = w_{11ij} = 0$$

$$w_{120} = P_0 + N_2 - N_1^T A_{c0}, \quad w_{12i} = P_i + N_2 - N_1^T A_{ci}$$

$$w_{12ij} = N_2 - N_1^T A_{cij}$$

$$w_{220} = \sum_{k=1}^{p} P_k \bar{\dot{\theta}}_k - N_2^T A_{c0} - A_{c0}^T N_2 + Q + C^T F_0^T R F_0 C$$

$$w_{22i} = -N_2^T A_{ci} - A_{ci}^T N_2 + C^T F_0^T R F_i C + C^T F_i^T R F_0 C$$

$$w_{22ij} = -N_2^T A_{cij} - A_{cij}^T N_2 + C^T F_i^T R F_j C + C^T F_j^T R F_i C$$

$$A_{c0} = A_0 + B_0 F_0 C, \quad A_{ci} = A_i + B_0 F_i C + B_i F_0 C$$

$$A_{cij} = B_i F_j C$$

Proof
For the first derivative of the Lyapunov function $V(\theta) = x(t)^T P(\theta) x(t)$ one obtains

$$\frac{dV(\theta)}{dt} = \dot{x}(t)^T P(\theta) x(t) + x(t)^T P(\dot{\theta}) x(t) + x(t)^T P(\theta)\dot{x}(t) =$$

$$[\dot{x}(t)^T \quad x(t)^T] \begin{bmatrix} 0 & P(\theta) \\ P(\theta) & P(\dot{\theta}) \end{bmatrix} [\dot{x}(t)^T \quad x(t)^T]^T$$

where

$$P(\dot{\theta}) = \sum_{i=1}^{p} P_i\dot{\theta}_i \leq \sum_{i=1}^{p} P_i \bar{\dot{\theta}}_i$$

assuming $P_i > 0$, $i = 1, 2 \ldots, p$. For closed-loop system (9.4) one can write $(x(t) = x)$

$$[\dot{x}^T N_1^T + x^T N_2^T][\dot{x} - A_c(\theta)x] + ([\dot{x}^T N_1^T + x^T N_2^T][\dot{x} - A_c(\theta)x])^T = 0 \tag{9.20}$$

where $N_1, N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices. Summarizing the above two equations, for the first derivative of the Lyapunov function it holds

$$\frac{dV(\theta)}{dt} = [\dot{x}^T \quad x^T] \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} \end{bmatrix} [\dot{x}^T \quad x^T]^T \tag{9.21}$$

where

$$u_{11} = N_1^T + N_1, \quad u_{12} = P(\theta) + N_2 - N_1^T A_c(\theta),$$

$$u_{22} = P(\dot{\theta}) - N_2^T A_c(\theta) - A_c(\theta)^T N_2$$

Substituting control algorithm (9.3) to system performance (9.5) for $J(t)$ one obtains

$$J(t) = x^T(Q + C^T F(\theta)^T R F(\theta) C)x \tag{9.22}$$

Rewriting the Bellman-Lyapunov equation (9.9) to the form

$$B_e = \left\{ \frac{dV(\theta)}{dt} + J(t) \right\} \leq 0 \tag{9.23}$$

or using equations (9.21) and (9.22) for B_e it holds

$$B_e = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}^T \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} + Q + C^T F(\theta)^T R F(\theta) C \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{9.24}$$

Inequalities (9.23), (9.9) hold if the inner matrix of (9.24) is negative definite (semidefinite), that is

$$W(\theta) = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} + Q + C^T F(\theta)^T R F(\theta) C \end{bmatrix} \leq 0 \tag{9.25}$$

Due to the quadratic function of $W(\theta)$ with respect to θ, *Lemma 9.2* gives the stability conditions in the form of (9.19) which proves the sufficient stability conditions of *Theorem 9.2*.
For the switched controller design procedure the last term of $W(\theta)$ needs to be symmetrized as follows

$$\sum_{i=1}^{p}\sum_{j=1}^{p} W_{ij}\theta_i\theta_j = \sum_{i=1}^{p} W_{ii}\theta_i^2 + \tag{9.26}$$

$$\sum_{i=1}^{p}\sum_{j>i}^{p}(W_{ij} + W_{ji})\theta_i\theta_j$$

where

$$W_{ii} = \begin{bmatrix} 0 & -N_1^T A_{cii} \\ A_{cii}^T N_1 & -N_2^T A_{cii} - A_{cii}^T N_2 + C^T F_i^T R F_i C \end{bmatrix} \tag{9.27}$$

Remarks:

1. Note that the dwell-time determines the minimal time interval between switching. If the real switching time interval is greater, $T \geq T_d$, the switched controller design procedure for arbitrarily switching proposed in [5, 6] becomes more complicated and conservative. Previously consideration imply that for arbitrarily switching algorithm the number of active switching plant mode generate the value of switching variable σ to determine which controller will be active. Switched controllers parameters calculation are made off-line by minimizing a given performance. Dwell-time switching is another switching algorithm which in real time (on-line) "t" requires an algorithm capable of minimizing some performance like output estimation error over the switching controllers. As a results of minimization dwell-time

switching logic generate the switching variable σ to determine which controller will be active in time "t" [9].

2. To obtain a feasible solution, in the switched controller design procedure proposed in the paper one can use a free and open source BMI solver Penlab.

9.4 Examples

The first example is taken from [16]. This plant has been constructed to include the technical challenge of the control of models in practice such as models like the air induction system of a turbocharged diesel engine. After a small simplification one obtains a simple linear time-varying plant with parameter varying coefficients

$$\begin{aligned} \dot{x}(t) &= a(\alpha)x(t) + b(\alpha)u(t) \\ y(t) &= x(t) \end{aligned} \tag{9.28}$$

where $\alpha(t) \in \mathbb{R}$ is an exogenous signal that changes the parameters of the plant as follows

$$\begin{aligned} a(\alpha) &= -6 - \frac{2}{\pi}\arctan\left(\frac{\alpha}{20}\right) \\ b(\alpha) &= \frac{1}{2} + \frac{5}{\pi}\arctan\left(\frac{\alpha}{20}\right) \end{aligned} \tag{9.29}$$

Let the problem be to design a switched PI controller which will guarantee the closed-loop stability and guaranteed cost for system (9.28) where $\alpha(t)$ is changing in steps between 0, 30 and 100. In these working points the calculated transfer functions are as follows

$$\begin{aligned} G_{s1}|_{\alpha=0} &= \tfrac{0.5}{s+6}, \quad G_{s2}|_{\alpha=30} = \tfrac{2.064}{s+6.626} \\ G_{s3}|_{\alpha=100} &= \tfrac{2.686}{s+6.874}, \end{aligned} \tag{9.30}$$

We transform the above transfer functions to the time domain to obtain a gain scheduling model in the form (9.1). Matrices A_i, B_i, $i = 0, 1, 2, \ldots, p$ for the case $i = 0$ calculated as middle values of all corresponding matrices and for $i = 1, 2, \ldots, p$ one can use the standard approach. The obtained model is extended by one state variable for PI controller design. The extended model is given as follows

$$\begin{aligned} A_0 &= \begin{bmatrix} -6.5 & 0 \\ 1 & 0 \end{bmatrix}, & A_1 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -0.126 & 0 \\ 0 & 0 \end{bmatrix}, & A_3 &= \begin{bmatrix} -0.374 & 0 \\ 0 & 0 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 1.75 \\ 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} -1.25 \\ 0 \end{bmatrix}, \end{aligned}$$

$$B_2 = \begin{bmatrix} 0.314 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.936 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0$$

Using *Theorems 9.1* and *9.2* with weighting matrices $Q = qI$, $q = 1 \times 10^{-6}$, $R = rI$, $r = 1$ and $\rho_U = 1 \times 10^7$, $\rho_L = 1 \times 10^{-5}$ which are the upper/lower constraint of the Lyapunov matrix ($\rho_L I < P(\theta) < \rho_U I$), $\bar{\dot{\theta}}_i = 1000$ $[1/s]$ is the maximal rate of switched parameter change for multi-quadratic stability approach and with $\theta \in \langle 0, 1 \rangle$ we obtain switched controllers in the form (9.3) where for the case of quadratic stability (*Theorem 9.1*) one has

$$\begin{aligned} F_0 &= [\ -0.8213 \quad -5.2172\ ] = [-0.8213 - \tfrac{-5.2172}{s}] \\ F_1 &= [\ -0.4868 \quad -3.0799\ ] \\ F_2 &= [\ -2.6625 \quad -16.8453\ ] \\ F_3 &= [\ \ 0.8072 \quad 5.1071\ ] \end{aligned} \tag{9.31}$$

and for the case of multi-quadratic stability approach (*Theorem 9.2*) one has

$$\begin{aligned} F_0 &= [\ -0.3412 \quad -10.9588\ ] \\ F_1 &= [\ -0.0475 \quad -0.3355\ ] \times 10^{-8} \\ F_2 &= [\ \ 0.0186 \quad 0.1315\ ] \times 10^{-7} \\ F_3 &= [\ \ 0.0637 \quad 0.4492\ ] \times 10^{-8} \end{aligned} \tag{9.32}$$

In simulations α will be switched between values 0, 30 and 100 and a particular arbitrarily switching algorithm in simulations is shown in *Figs. 9.3* and *9.5*. The switched parameters are calculated from signal α and switched with a maximal rate of change $\bar{\dot{\theta}}_i$. Simulation results (*Figs. 9.1*, *9.2* and *9.3*) confirm that *Theorems 9.1* and *9.2* hold, thus the closed loop switched system is stable for a prescribed rate of switching signal change.

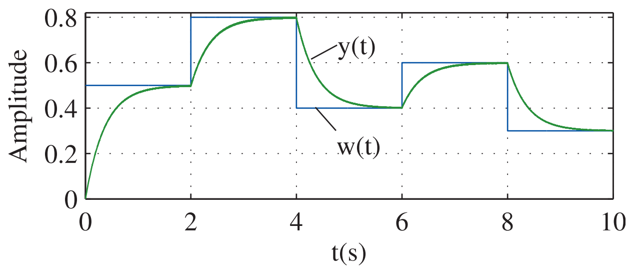

FIGURE 9.1: Simulation results $w(t)$, $y(t)$ with switched controller (9.31) – QS

Other switched controllers are designed using *Theorems 9.1* and *9.2* with weighting matrices $Q = qI$, $q = 10$, $R = rI$, $r = 1$, $\rho_U = 1 \times 10^7$, $\rho_L = 1 \times 10^{-5}$, $\bar{\dot{\theta}}_i = 2000$ $[1/s]$ is the maximal rate of switched parameters change for multi-quadratic stability approach and with $\theta \in \langle 0, 1 \rangle$. The obtained controllers are in the form (9.3), where for the case of quadratic stability (*Theorem 9.2*) one has

$$BMI\ solver\ failed.$$

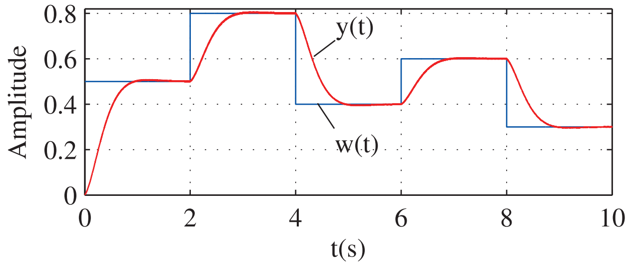

FIGURE 9.2: Simulation results $w(t)$, $y(t)$ with switched controller (9.32) – MPQS

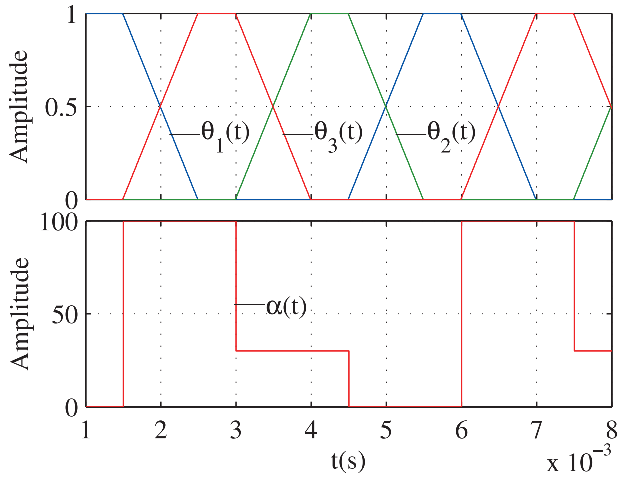

FIGURE 9.3: Calculated switched parameters $\theta(t)$ and the switching signal $\alpha(t)$ for the case of $\bar{\dot{\theta}}_i = 1000\ [1/s]$

and for the case of multi-quadratic stability approach (*Theorem 9.2*) one has

$$\begin{aligned} F_0 &= [\ -540.8653\ \ -1.3937\ ] \\ F_1 &= [\ \ -0.2596\ \ -0.0003\ ] \times 10^{-10} \\ F_2 &= [\ \ 0.1033\ \ \ 0.0001\ ] \times 10^{-9} \\ F_3 &= [\ \ 0.3467\ \ \ 0.0005\ ] \times 10^{-10} \end{aligned} \tag{9.33}$$

For this case, in simulations α will be switched between the same values 0, 30 and 100 as shown in *Fig. 9.5*. The switched parameters are calculated from this signal and switched with maximal rate of change $\bar{\dot{\theta}}_i = 2000\ [1/s]$. Simulation results (*Figs. 9.4*, *9.5*) confirm that the multi-quadratic stability approach is less conservative than the quadratic stability approach. This example implies that for higher values of weighting matrix $Q = qI$, $q > 0.1$ (higher weight for quality) quadratic stability with BMI solver fails but with the multi-quadratic stability approach we can obtain the controller up to $q = 15$.

The second example is borrowed from [17]. Consider a simplified manual transmission model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= [-a(x_2)/v + u]/(1 + v) \end{aligned} \tag{9.34}$$

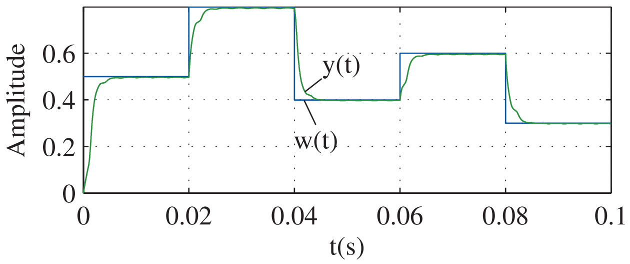

FIGURE 9.4: Simulation results $w(t)$, $y(t)$ with switched controller (9.33) – MPQS

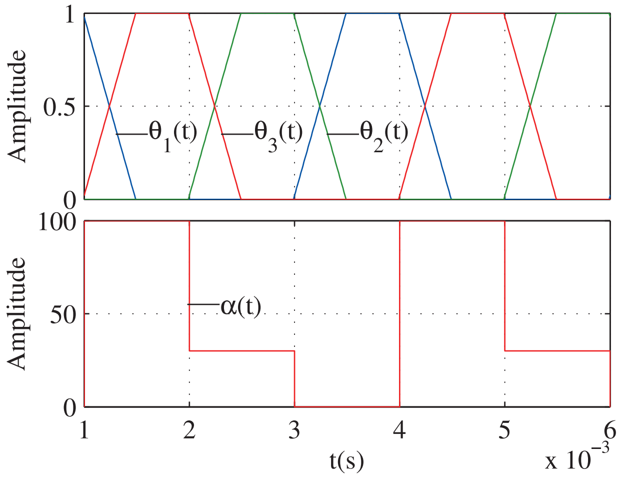

FIGURE 9.5: Calculated switching parameters $\theta(t)$ and the switching signal $\alpha(t)$ for the case of $\bar{\dot{\theta}}_i = 2000$ $[1/s]$

where x_1 is the ground speed, x_2 is the acceleration, $u \in \langle 0, 1\rangle$ is the throttle position, and $v \in \{1, 2, 3, 4\}$ is the gear shift position. Function $a(.)$ is positive for a positive argument. Model (9.34) can be transformed to this form

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \frac{-a}{v+v^2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{1+v} \end{bmatrix}}_{B} u \tag{9.35}$$

Substituting $a = 5$ and $v = [1, 2, 3, 4]$ we can transform (9.35) to the form (9.1)

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, & A_1 &= \begin{bmatrix} 0 & 0 \\ 0 & -1.5 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.1667 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.5833 \end{bmatrix} \\ A_4 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.75 \end{bmatrix}, & B_0 &= \begin{bmatrix} 0 \\ 0.3208 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ 0.1792 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 \\ 0.0125 \end{bmatrix} \end{aligned}$$

$$B_3 = \begin{bmatrix} 0 \\ -0.0708 \end{bmatrix}, \; B_4 = \begin{bmatrix} 0 \\ -0.1208 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1.75 \end{bmatrix}$$

Using *Theorems* 9.1 and 9.2 with weighting matrices $Q = qI$, $q = 0.1$, $R = rI$, $r = 208$ and $\rho_U = 1 \times 10^5$, $\rho_L = 1 \times 10^{-5}$, $\bar{\dot{\theta}}_i = 1000$ $[1/s]$ we obtain switched controller in the form (9.3) where for the case of quadratic stability (*Theorem* 9.1) one has

$$BMI\ solver\ failed. \tag{9.36}$$

and for the case of multi-quadratic stability approach (*Theorem* 9.2) one has

$$\begin{aligned} F_0 &= [\ \ 24.0495 \quad 76.9511\ ] \\ F_1 &= [\ -25.1832 \ -80.2151\ ] \\ F_2 &= [\ -24.8145 \ -79.0407\ ] \\ F_3 &= [\ -24.6836 \ -78.6237\ ] \\ F_4 &= [\ -24.5773 \ -78.2853\ ] \end{aligned} \tag{9.37}$$

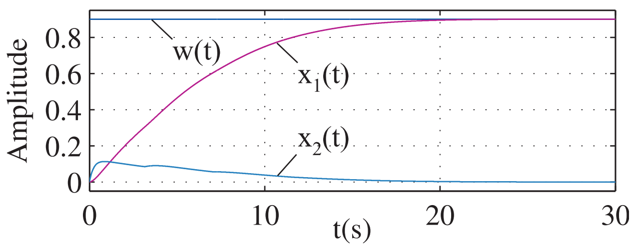

FIGURE 9.6: Simulation results $w(t)$, $y(t)$ with switched controller (9.37) – MPQS

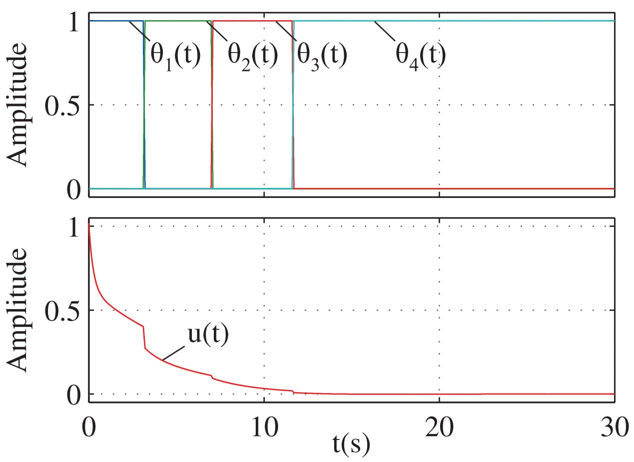

FIGURE 9.7: Calculated switching parameters $\theta(t)$ and the controller output with switched controller (9.37) – MPQS

In the simulations we switched the gear shift as follows $v = 1$ if $x_1 \in \langle 0, 0.3\rangle$, $v = 2$ if $x_1 \in (0.3, 0.6\rangle$, $v = 3$ if $x_1 \in (0.6, 0.8\rangle$ and $v = 4$ if $x_1 \in (0.8, \infty)$, and the switching rate of v is established with $\bar{\dot{\theta}}_i = 10$ $[1/s]$. From the simulation results (*Figs.* 9.6 and 9.7) it follows that the theorems holds and that the multi-parameter quadratic stability approach is less conservative than the quadratic stability and that with the weighting matrices we can affect the performance quality and tune the system to the desired conditions.

Third example. Control systems over data networks are commonly referred to as networked control systems (NCSs). For the NCSs, the sampled data and controller signals are transmitted through a network. As a result, it leads to a network-induced delay in a networked control closed-loop system. The existence of such a kind of delay in a network-based control loop can induce instability or poor performance of control systems. Assume that a linear system with transfer function $G(s)$ is integrated to NCSs which inevitably leads to a change of the plant transfer function as $G(s)e^{-T_d s}$, where T_d is a variable plant time delay. The value of T_d depends on the load of the communication network. Assume that for four communication network loads one can define four middle values of time delays T_{di}, $i = 1, 2, 3, 4$.

For PI switched controller design and simulation we will use a laboratory model of a DC-motor which is one of the real processes built for control education and research at our institute. We have identified the DC motor system and the following transfer function has been obtained

$$Sys = \frac{0.0627s + 1.281}{2.081s^2 + 2.506s + 1} \tag{9.38}$$

For the defined 4 middle values of the induced time delays $T_d = [0.1,\ 0.2,\ 0.3,\ 0.4]\, s$ and using the first order Pade approximation we computed 4 plant transfer functions which are transformed to the state space. The obtained 4 plant models are extended with one state for the switched PI controller design. Finally one obtains the plant models in the form (9.1)

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5.0006 & -13.0218 & -11.6208 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4.6005 & -11.5382 & -9.5832 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2001 & 0.5018 & 0.4168 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.8001 & 4.5147 & 3.7498 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.6003 & 6.5218 & 5.4168 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad B_0 = \begin{bmatrix} -0.0308 \\ 0.0498 \\ 4.1652 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.0002 \\ 0.5833 \\ -4.5362 \\ 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -0.0002 \\ -0.0261 \\ 2.1004 \\ 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.0008 \\ -0.2288 \\ 1.5999 \\ 0 \end{bmatrix},\quad B_4 = \begin{bmatrix} -0.0003 \\ -0.3284 \\ 0.8360 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For switched controller design, *Theorem 9.2* with weighting matrices $Q = qI$, $q = 0.01$, $R = rI$, $r = 2$ and $\rho_U = 10$, $\rho_L = 1 \times 10^{-5}$, $\bar{\dot{\theta}}_i = 10$ $[1/s]$ will be used. The obtained switched PI controller is the form (9.3)

The case of multi-quadratic stability approach (Theorem 9.2)

$$
\begin{aligned}
F_0 &= [\ -0.3296\ \ -0.1352\ ] \\
F_1 &= [\ -0.4460\ \ -0.1981\ ] \\
F_2 &= [\ -0.0589\ \ -0.0293\ ] \\
F_3 &= [\ -0.0217\ \ -0.0106\ ] \\
F_4 &= [\ \ \ 0.1439\ \ \ \ 0.0908\ ] \times 10^{-10}
\end{aligned} \tag{9.39}
$$

Simulation results (*Figs. 9.8*, *9.9* and *9.10*) confirm, that *Theorem 9.2* holds. In the simulation the switching algorithm (middle time delay) is shown in *Fig. 9.10* from which the scheduled parameters are calculated.

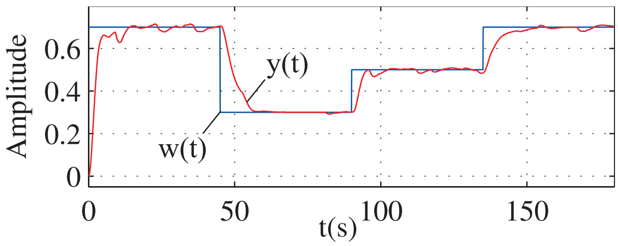

FIGURE 9.8: Simulation results $w(t)$, $y(t)$ with switched controller (9.39)

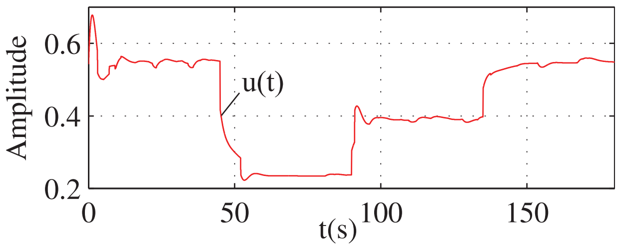

FIGURE 9.9: Switched Controller output (9.39)

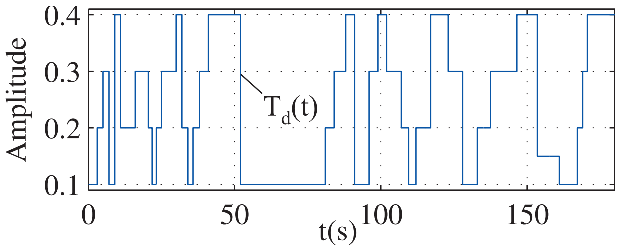

FIGURE 9.10: Time delay changes

9.5 Conclusion

The paper addresses the problem of the switched controller design with arbitrarily switching algorithm which ensures the closed-loop stability and guaranteed cost for a prescribed rate of change of system switching. A novel gain scheduling plant model is presented for linear continuous-time invariant switched systems. The first proposed method is connected with the notion of quadratic stability with respect to switched parameter θ. In this case we assume that the rate of θ change is infinite. In some real cases the rate of change of the switching signal is finite. This assumption was used in the second approach to obtain the switched controller design procedure. The advantages of the proposed method are:

- one can obtain less conservative results with respect to using the dwell-time approach,
- for the switched controller design there is no need to use the approach of the "dwell-time" which markedly complicates the design procedure,
- the rate of the switching signal change can be prescribed by the designer which opens the new possibilities for practical realizations and development of new theoretical approaches,
- the obtained design procedure for output/state feedback ensures the closed loop stability of switched systems and guaranteed cost,
- the obtained design procedure can be implemented easily to the standard LMI or BMI approaches,
- the obtained design procedure can be easily transformed to the case of robust switched controller design for continuous-time switched systems with arbitrarily switching.

Numerical examples illustrate the effectiveness of the proposed approach.

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10

Robust Switched Controller Design for Nonlinear Continuous Systems (Paper 7)

Abstract

A novel approach is presented to robust switched controller design for nonlinear continuous-time systems under an arbitrary switching signal using the gain scheduling approach. The proposed design procedure is based on the robust multi parameter dependent quadratic stability condition. The obtained switched controller design procedure for nonlinear continuous-time systems is in at bilinear matrix form (BMI). The properties of the obtained design are illustrated on simulation examples.

Keywords: Robust controller, Switched controller, Gain-scheduled controller, Continuous time Nonlinear systems, Lyapunov function.

10.1 Introduction

Switched systems play an important role in the past decade. Motivation for studying switched systems stems from two facts

- switched systems have numerous application in the control of real plants, and
- in real control, there are dynamical systems that cannot be stabilized by any continuous static output/state feedback control law, but a stabilizing switching control scheme can be found.

Therefore switched system stability and the controller design procedure are the most important issues, especially for nonlinear systems. Stability under arbitrary switching is guaranteed by the existence of a common Lyapunov function for all switching subsystems, [1], [2], [3], [4], [5], [6]. For switched linear systems, finding the common Lyapunov function is relatively easy but for nonlinear systems it is difficult. A survey of switched controller design for nonlinear systems can be found in [1], [2], [3], [7] and references therein. Due to the switched controller design problems for continuous time nonlinear systems in this paper we pursue the idea to use, instead of a nonlinear switched plant model a switched gain-scheduled plant model and to design the robust switched gain-scheduled controller guaranteeing closed-loop stability and guaranteed cost for all operating points of the switched nonlinear system.

Gain scheduling control deals with systems subject to parametric variations, which include linear systems with time-varying parameters or nonlinear systems modeled by a family of linear parameter-varying systems, [8]. Because of time-varying gain-scheduled parameters many researches have tackled the design problem of the gain-scheduled controllers for linear time-varying system (LPV) using linear matrix inequalities (LMI) and the Lyapunov function approach [8], [9], [10], [11]. Reviews of the gain-scheduled controller design can be found in [12], [13].

In this paper a method is proposed of robust switched controller design for nonlinear continuous time switched systems using the gain scheduling approach. There are only some results in the field of continuous time switched gain-scheduled controller design for continuous time systems [14], [15], [16] and stabilization of switched continuous time linear systems [17], [18]. Representative are two last references [17], [18]. In the first paper the authors introduce the notion of the dwell-time T_d (minimal time interval between switching) into the switched controller design procedure. In the stability analysis condition of switched systems the dwell-time is included to the term $e^{A_c T_d}$ (A_c-closed-loop matrix (10.4)). In such a way the proposed design procedure for stability analysis and switching controller design for the real switching time interval $T > T_d$ becomes conservative. The "dwell-time term" for continuous-time systems very complicated the switched controller design procedure. In the paper [18] sufficient conditions are given for the stability of linear systems with dwell-time and with polytopic type parameter uncertainty. Lyapunov functions, in quadratic form for each mode, which are non-increasing at the switching instants are assigned to each mode. During the dwell-time this function varies piecewise linearly in time after switching occurs. The proposed method was applied to stabilization via a state feedback both for nominal and uncertain cases. Since within the dwell-time the Lyapunov function varies piecewise linearly and the real switching time interval $T > T_d$, the switching controller design procedure become rather conservative. The switched controller design procedure for continuous-time systems proposed in this paper does not use the approach of the "dwell-time", therefore there is no such drawback as mentioned in the above references.

Vast references on the switched controller design are concentrated on the case where switching can occur immediately (ideal case), for a large number of switched system the realistic case is where the rate of change of the switching signal is finite (non-ideal case). A assumption of a non-ideal case of the switching variable will be used in this paper which gives other opportunities for controller designer. Some results about the stability of a class of uncertain linear varying systems (transform to switched systems) can be found in [19].

The remainder of the paper is organized as follows. In *Section 10.1* and *10.2* we present the class of switched gain-scheduled control systems. In *Section 10.3* we address the output feedback PI switched gain-scheduled robust controller design procedure for continuous time gain-scheduled plant model. Finally, in *Section 10.4*, the proposed design procedure is demonstrated on a simple example.

Our notation is standard, $P \in \mathbb{R}^{m\times n}$ denotes the set of real $m \times n$ matrices, $P > 0$ $(P \geq 0) \in \mathbb{R}^{n\times n}$ is a real symmetric, positive definite (semidefinite) matrix. $\sigma \in S$ indicates the arbitrary switching algorithm and $\sigma + 1$ is the first next mode to mode σ. "*" in matrices denotes the respective transposed (conjugate) term to make matrix symmetric. I_m is an $m \times m$ identity matrix, 0_m denotes the zero matrix.

10.2 Problem statement and preliminaries

10.2.1 Uncertain LPV plant model for switched systems

Consider family of nonlinear switched systems

$$\begin{aligned} \dot{z} &= f_\sigma(z, v, w) \quad \sigma \in S = \{1, 2, \ldots, N\} \\ \overline{y} &= h(z) \end{aligned} \tag{10.1}$$

where $z \in \mathbb{R}^n$ is the state, the input $v \in \mathbb{R}^m$, the output $\overline{y} \in \mathbb{R}^l$, exogenous input $w \in \mathbb{R}^k$ which captures parametric dependence of the plant (10.1) on exogenous input. The arbitrary switching algorithm $\sigma \in S$ is a piecewise constant, right continuous function which specifies at each time the index of the active system, [20]. Assume that $f(.)$ is locally Lipschitz for every $\sigma \in S$. Consider that the number of equilibrium points for each switching modes is equal to p, that is for each mode $\sigma \in S$ the nonlinear system can be replaced by a family of p linearized plant. For more details how to obtain the gain-scheduled plant model see excellent surveys [12], [13]. To receive the model uncertainty of the gain-scheduled plant it is necessary to obtain other family of linearized plant models around the p equilibrium points. Finally, one obtains the gain-scheduled uncertain plant model in the form

$$\begin{aligned} \dot{x} &= \overline{A}_\sigma(\xi, \theta)x + \overline{B}_\sigma(\xi, \theta)u \quad \sigma \in S \\ y &= Cx \end{aligned} \tag{10.2}$$

where $x = z - z_e$, $u = v - v_e$, $y = \overline{y} - \overline{y}_e$, $(z_e, v_e, \overline{y}_e)$ define the equilibrium family for plant (10.1). Assume, that for $i - th$ equilibrium point one obtain the sets $x \in X_i$, $u \in U_i$, $y \in Y_i$, $i = 1, 2, \ldots, p$. Summarizing above sets we get $x \in X = \bigcup_{i=1}^{p} X_i, u \in$

$U = \bigcup_{i=1}^{p} U_i, y \in Y = \bigcup_{i=1}^{p} Y_i$.

$$\begin{aligned} \overline{A}_\sigma(\xi,\theta) &= A_{\sigma 0}(\xi) + \sum_{j=1}^{p} A_{\sigma j}(\xi)\theta_j \in \mathbb{R}^{n\times n} \\ \overline{B}_\sigma(\xi,\theta) &= B_{\sigma 0}(\xi) + \sum_{j=1}^{p} B_{\sigma j}(\xi)\theta_j \in \mathbb{R}^{n\times m} \end{aligned} \tag{10.3}$$

Matrices $A_{\sigma j}(\xi)$, $B_{\sigma j}(\xi)$, $j = 0, 1, 2, \ldots, p$ belong to the convex set a polytope with K vertices that can formally defined as

$$\begin{aligned} \Omega_\sigma = \Bigg\{ & A_{\sigma j}(\xi), B_{\sigma j}(\xi) = \sum_{i=1}^{K} (A_{\sigma ij}, B_{\sigma ij})\, \xi_i \\ & j = 0, 1, 2, 3, \ldots, p, \sum_{i=1}^{K} \xi_i = 1, \xi_i \geq 0, \xi_i \in \Omega_\xi \Bigg\} \end{aligned} \tag{10.4}$$

where ξ_i, $i = 1, 2, \ldots, K$ are constant or possible time-varying but unknown parameters, $A_{\sigma ij}$, $B_{\sigma ij}$, C are constant matrices of corresponding dimensions, $\theta \in \mathbb{R}^p$ is a vector of known constant or time-varying gain-scheduled parameter. Assume that both lower and upper bounds are available, that is

$$\begin{aligned} \theta &\in \Omega_s = \{\theta \in \mathbb{R}^p : \theta_j \in \langle \underline{\theta}_j, \overline{\theta}_j \rangle\} \\ \dot{\theta} &\in \Omega_t = \{\dot{\theta} \in \mathbb{R}^p : \dot{\theta}_j \in \langle \underline{\dot{\theta}}_j, \overline{\dot{\theta}}_j \rangle\} \end{aligned} \tag{10.5}$$

10.2.2 Problem formulation

For each plant mode consider the uncertain gain-scheduled LPV plant model in the form (10.2), (10.3) and (10.4)

$$\begin{aligned} \dot{x} &= \left(A_{\sigma 0}(\xi) + \sum_{j=1}^{p} A_{\sigma j}(\xi)\theta_j \right) x + \left(B_{\sigma 0}(\xi) + \sum_{j=1}^{p} B_{\sigma j}(\xi)\theta_j \right) u \\ y &= Cx \end{aligned} \tag{10.6}$$

For a robust gain-scheduled I part controller design, the states x of (10.6) need to be extended in such a way that a static output feedback control algorithm can provide proportional (P) and integral (I) parts of the designed controller, for more detail see [21]. Assume that system (10.6) allows PI controller design with a static output feedback. The feedback control law is considered in the form

$$u = F_\sigma(\theta) y = \left(F_{\sigma 0} + \sum_{j=1}^{p} F_{\sigma j}\theta_j \right) Cx \tag{10.7}$$

where $F_\sigma(\theta)$ is the static output feedback gain-scheduled controller for mode σ. The closed-loop system is

$$\dot{x} = A_{\sigma c}(\xi, \theta, \alpha) x \tag{10.8}$$

where

$$A_{\sigma c}(\xi,\theta,\alpha) =$$

$$\sum_{\sigma=1}^{N}\left(\overline{A}_{\sigma}(\xi,\theta)+\overline{B}_{\sigma}(\xi,\theta)F_{\sigma}(\theta)C\right)\alpha_{\sigma} = \sum_{\sigma=1}^{N} A_{\sigma}(\xi,\theta)\alpha_{\sigma}$$

$$\alpha^{T} = [\alpha_1,\alpha_2,\ldots,\alpha_N], \sum_{\sigma=1}^{N}\alpha_{\sigma} = 1, \sum_{\sigma=1}^{N}\dot{\alpha}_{\sigma} = 0$$

$\alpha_j = 1$ when σ_j is active plant mode, else $\alpha_j = 0$. Assume $\alpha \in \Omega_{\alpha}$, $\dot{\alpha} \in \Omega_d$.

To access the system performance, we consider an original weighted scheduled quadratic cost function

$$J = \int_{t=0}^{\infty} J(t)dt \tag{10.9}$$

where $J(t) = x^T Q(\theta)x + u^T Ru$ and

$$Q(\theta) = Q_0 + \sum_{j=1}^{p} Q_j\theta_j, \quad Q_j \geq 0, \quad R > 0$$

.

Definition 10.1. Consider a stable closed-loop switched system (10.8) with N modes. If there is a control algorithm (10.7) and a positive scalar J^* such that the closed-loop cost function (10.9) satisfies $J \leq J^*$ for all $\theta \in \Omega_s$, $\alpha \in \Omega_{\alpha}$, then J^* is said to be a guaranteed cost and "u" is said to be a guaranteed cost control algorithm for arbitrary switching algorithm $\sigma \in S$.

Theorem 10.1. *[22] Control algorithm (10.7) is the guaranteed cost control law for the switched closed-loop system (10.8) if and only if there is Lyapunov function $V(x,\xi,\theta,\alpha) > 0$, matrices $Q(\theta)$, R and gain matrices $F_{\sigma k}$, $k = 0,1,\ldots,p$ such that for $\sigma \in S$ the following inequality holds*

$$B_e = \frac{dV(x,\xi,\theta,\alpha)}{dt} + J(t) \leq -\varepsilon x^T x, \varepsilon \to 0 \tag{10.10}$$

10.3 Main results

This section formulates the theoretical approach to the robust switched gain-scheduled controller design with control law (10.7) which ensure closed-loop multi parameter dependent quadratic stability and guaranteed cost for an arbitrary switching algorithm $\sigma \in S$. Assume that in Theorem 10.1 the Lyapunov function is in the form

$$V(x,\xi,\theta,\alpha) = x^T P(\xi,\theta,\alpha)x \tag{10.11}$$

where the Lyapunov multi parameter-dependent matrix is

$$P(\xi,\theta,\alpha) = \sum_{i=1}^{K}\left(P_{0i} + \sum_{\sigma=1}^{N}\left(P_{\sigma 0i} + \sum_{j=1}^{p} P_{\sigma ij}\theta_j\right)\alpha_{\sigma}\right)\xi_i \tag{10.12}$$

Time derivative of the Lyapunov function(10.11) is

$$\dot{V}(.) = [\dot{x}^T \quad x^T] \begin{bmatrix} 0 & P(\xi,\theta,\alpha) \\ P(\xi,\theta,\alpha) & \dot{P}(\xi,\theta,\alpha) \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{10.13}$$

where

$$\dot{P}(.) = \sum_{i=1}^{K} \sum_{\sigma=1}^{N} DP_{\sigma i} \alpha_\sigma \xi_i \tag{10.14}$$

$$DP_{\sigma i} = \sum_{\sigma=1}^{N} P_{\sigma 0 i} \dot{\alpha}_\sigma + \sum_{j=1}^{p} P_{\sigma ij} \dot{\theta}_j + \sum_{j=1}^{p} \sum_{\sigma=1}^{N} P_{\sigma ij} \dot{\alpha}_\sigma \theta_j$$

Using equality

$$(2N_1\dot{x} + 2N_2 x)^T \left(\dot{x} - \sum_{\sigma=1}^{N} A_\sigma(\xi,\theta)\alpha_\sigma x \right) = 0 \tag{10.15}$$

equation (10.13) can be rewritten as

$$\frac{dV(.)}{dt} = \sum_{\sigma=1}^{N} \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix} L_\sigma(\xi,\theta) \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{10.16}$$

$$L_\sigma(\xi,\theta) = \{l_\sigma(i,j)\}_{2\times 2}$$

$$l_\sigma(1,1) = N_1^T + N_1$$

$$l_\sigma(1,2) = -N_1^T A_\sigma(\xi,\theta) + N_2 + \sum_{i=1}^{K} \left(P_{0i} + P_{\sigma 0 i} + \sum_{j=1}^{p} P_{\sigma ij}\theta_j \right) \xi_i$$

$$l_\sigma(2,2) = -N_2^T A_\sigma(\xi,\theta) - A_\sigma^T(\xi,\theta) N_2 + \sum_{i=1}^{K} DP_{\sigma i}\xi_i$$

where $N_1,\ N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices.

On substituting (10.7) to (10.9) one obtains

$$J(t) = x^T S(\theta) x \tag{10.17}$$

where

$$S(\theta) = S_0 + \sum_{j=1}^{p} S_j\theta_j + \sum_{j=1}^{p} \sum_{k>j}^{p} S_{kj}\theta_k\theta_j + \sum_{k=1}^{p} S_{kk}\theta_k^2$$

$$S_0 = Q_0 + C^T F_{\sigma 0}^T R F_{\sigma o} C$$

$$S_j = Q_j + C^T \left(F_{\sigma 0}^T R F_{\sigma_j} + F_{\sigma j}^T R F_{\sigma 0} \right) C$$

$$S_{jk} = C^T \left(F_{\sigma j}^T R F_{\sigma k} + F_{\sigma k}^T R F_{\sigma j} \right) C$$

$$S_{kk} = C^T F_{\sigma k}^T R F_{\sigma k} C$$

The switched model plant (10.8) can be rewritten to the form

$$A_\sigma(\xi,\theta) = M_0(\xi) + \sum_{j=1}^{p} M_j(\xi)\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} M_{jk}(\xi)\theta_j\theta_k + \sum_{k}^{p} M_{kk}(\xi)\theta^2 \tag{10.18}$$

where

$$M_0(\xi) = A_{\sigma 0}(\xi) + B_{\sigma 0}(\xi)F_{\sigma 0}C$$
$$M_j(\xi) = A_{\sigma j} + (B_{\sigma 0}(\xi)F_{\sigma j} + B_{\sigma j}(\xi)F_{\sigma 0})C$$
$$M_{jk}(\xi) = (B_{\sigma j}(\xi)F_{\sigma k} + B_{\sigma k}(\xi)F_{\sigma j})C$$
$$M_{kk}(\xi) = B_{\sigma k}(\xi)F_{\sigma k}C$$

Due to *Theorem 10.1* the closed-loop switched gain-scheduled system is multi parameter dependent quadratically stable with guaranteed cost for $\sigma \in S$, ξ_i, $i = 1, 2, \dots, K$ if the following inequalities hold

$$B_e = [\dot{x}^T \quad x^T]W(\xi,\sigma,\theta)[\dot{x}^T \quad x^T]^T \le 0 \tag{10.19}$$

where $W(\xi,\sigma,\theta) = \{w_{ij}(\sigma,\xi)\}_{2\times 2}$

$$\begin{aligned} w_{11}(\sigma,\xi) &= N_1^T + N_1 \\ w_{12}(\sigma,\xi) &= \sum_{i=1}^{K}\left(P_{0i} + P_{\sigma 0i} + \sum_{j=1}^{p} P_{\sigma ij}\theta_j\right)\xi_i \\ &\quad - N_1^T A_\sigma(\xi,\theta) + N_2 \\ w_{22}(\sigma,\xi) &= -N_2^T A_\sigma(\xi,\theta) - A_\sigma(\xi,\theta)^T N_2 \\ &\quad + \sum_{i=1}^{K} DP_{\sigma i}\xi_i + S(\theta) \end{aligned}$$

Inequality (10.19) implies :

- for all $\sigma \in S$ the inequality is linear with respect to uncertain parameter ξ_i, $i = 1, 2, \dots, K$,
- for all $\sigma \in S$ the inequality is a quadratic function with respect to the gain-scheduled parameters θ_i, $i = 1, 2, \dots, p$.

For the next development the following theorem is useful.

Theorem 10.2. *[23] Consider a scalar quadratic function of* $\theta \in \mathbb{R}^p$

$$f(\theta) = a_0 + \sum_{j=1}^{p} a_j\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} a_{jk}\theta_j\theta_k + \sum_{k}^{p} a_{kk}\theta_k^2 \tag{10.20}$$

and assume that if $f(\theta)$ is multiconvex, that is

$$\frac{\delta^2 f(\theta)}{\delta\theta_k^2} = 2a_{kk} \geq 0, k = 1, 2, \ldots, p$$

then $f(\theta)$ is negative definite in the hyper rectangle (10.5) if and only if it takes negative values at the vertices of (10.5), that is if and only if $f(\theta) < 0$ for all vertices of the set given by (10.5).

Due to (10.14), (10.17) and (10.18) the robust stability conditions of switched system can be rewritten as

$$\begin{aligned} W(\xi, \sigma, \theta) = \sum_{\sigma=1}^{N} L(\theta, \xi)\alpha_\sigma = \sum_{\sigma=1}^{N} (W_{\sigma 0}(\xi) + \\ + \sum_{j=1}^{p} W_{\sigma j}(\xi)\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} W_{\sigma jk}(\xi)\theta_j\theta_k + \\ + \sum_{k=1}^{p} W_{\sigma kk}\theta_k^2)\alpha_\sigma \leq 0 \end{aligned} \tag{10.21}$$

where $W_{\sigma 0}(\xi) = \{w_{0ij}^\sigma\}_{2\times 2}$, $W_{\sigma j}(\xi) = \{w_{jik}^\sigma\}_{2\times 2}$

$$\begin{aligned} w_{011}^\sigma &= N_1^T + N_1 \\ w_{012}^\sigma &= -N_1^T M_0(\xi) + N_2 + \textstyle\sum_{i=1}^K (P_{0i} + P_{\sigma 0i})\xi_i \\ w_{022}^\sigma &= -N_2^T M_0(\xi) - M_0^T(\xi) N_2 + S_0 + \\ &\quad + \textstyle\sum_{i=1}^K \left(P_{\sigma 0i}\dot{\alpha}_\sigma + \sum_{j=1}^p P_{\sigma ij}\dot{\theta}_j \right)\xi_i \\ w_{j11}^\sigma &= 0;\ w_{j12}^\sigma = -N_1^T M_j(\xi) + \textstyle\sum_{i=1}^K P_{\sigma ij}\xi_i \\ w_{j22}^\sigma &= -N_2^T M_j(\xi) - M_j(\xi)^T N_2 + S_j + \\ &\quad + \textstyle\sum_{i=1}^K \left(\sum_{\sigma=1}^N P_{\sigma ij}\dot{\alpha}_\sigma \right)\xi_i \end{aligned}$$

$$W_{\sigma jk}(\xi) = \begin{bmatrix} 0 & -N_1^T M_{jk}(\xi) \\ \text{"}*\text{"} & -N_2^T M_{jk}(\xi) - M_{jk}(\xi)^T N_2 + S_{jk} \end{bmatrix}$$

$$W_{\sigma kk}(\xi) = \begin{bmatrix} 0 & -N_1^T M_{kk}(\xi) \\ \text{"}*\text{"} & -N_2^T M_{kk}(\xi) - M_{kk}(\xi)^T N_2 + S_{kk} \end{bmatrix}$$

The main results on the robust stability condition for the switched gain-scheduled control system is given in the next theorem.

Theorem 10.3. *Closed-loop switched system (10.8) is robust multi parameter dependent quadratically stable with guaranteed cost if there is a positive definite matrix $P(\xi, \theta, \alpha) \in \mathbb{R}^{n\times n}$ (10.12), matrices N_1, $N_2 \in \mathbb{R}^{n\times n}$, positive definite (semidefinite) matrices $Q(\theta)$, R and gain-scheduled controller matrix $F_\sigma(\theta)$, such that for $\sigma \in S$*

1.

$$L_\sigma(\xi, \theta) < 0 \tag{10.22}$$

2.

$$W_{\sigma kk} \geq 0,\ \sigma \in S,\ \theta \in \Omega_s,\ k = 1, 2, \ldots, p$$

The proof of theorem sufficient conditions immediately follows from eqs. (10.12)-(10.21). Notes.

- $L_\sigma(\theta, \xi)$ is linear with respect to uncertain parameter ξ_i, $i = 1, 2, \ldots, K$, it holds $L_\sigma(\theta, \xi) = \sum_{i=1}^{K} L_{\sigma i}(\theta)\xi_i$, therefore inequality (10.22) for each $\sigma \in S$ split to K inequalities of type $L_{\sigma i}(\theta) < 0$ and $W_{\sigma ikk} \geq 0$.
- Eq. (10.12) implies that in dependence on the chosen structure of the Lyapunov matrix $P(\xi, \theta, \alpha)$ one should obtained different types of stability conditions from quadratic to multi parameter dependent quadratic stabilities. Different types of stability conditions determine the conservatism of the design procedure and the rate of change of corresponding variables.

10.4 Example

Consider a simple nonlinear switched system with two modes as

$$\begin{aligned} \sigma = 1, \dot{x} &= -asinx + bu \\ \sigma = 2, \dot{x} &= -acosx + bu, y = x \end{aligned} \tag{10.23}$$

where $a \in \langle 0.8, 1 \rangle$, when $a = 0.8$ then $b = 1$ and $a = 1$, $b = 0.5$. One can linearized model (10.23) in the three working points $x_0 = \{0, \pi/4, \pi/2\}$. For PI gain-scheduled controller design the system state space needs to be increased, finally for matrices $A(\sigma, \xi, \theta)$, $B(\sigma, \xi, \theta)$ one obtains

$$\begin{aligned} A(1, \xi, \theta) &= \left\{ \begin{bmatrix} -0.4 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.117 & 0 \\ 0 & 0 \end{bmatrix} \theta_1 + \begin{bmatrix} -0.28 & 0 \\ 0 & 0 \end{bmatrix} \theta_2 \right\} \xi_1 \\ &+ \left\{ \begin{bmatrix} -0.5 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.14645 & 0 \\ 0 & 0 \end{bmatrix} \theta_1 + \begin{bmatrix} -0.35355 & 0 \\ 0 & 0 \end{bmatrix} \theta_2 \right\} \xi_2 \\ B(1, \xi, \theta) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xi_1 + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \xi_2 \end{aligned} \tag{10.24}$$

$$\begin{aligned} A(2, \xi, \theta) &= \left\{ \begin{bmatrix} 0.4 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -0.287 & 0 \\ 0 & 0 \end{bmatrix} \theta_1 + \begin{bmatrix} -0.117 & 0 \\ 0 & 0 \end{bmatrix} \theta_2 \right\} \xi_1 \\ &+ \left\{ \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -.35955 & 0 \\ 0 & 0 \end{bmatrix} \theta_1 + \begin{bmatrix} -0.14645 & 0 \\ 0 & 0 \end{bmatrix} \theta_2 \right\} \xi_2 \\ B(2, \xi, \theta) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xi_1 + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \xi_2 \end{aligned} \tag{10.25}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For each mode, the robust gain-scheduled controller has been designed for the following parameters $Q_0 = 0.01 * I$, $Q_1 = 0.001 * I$, $Q_2 = 0.005 * I$, $R = I$, $\max \dot{\alpha}_\sigma = 100\frac{1}{s}$ (rate of switching parameters changing) $\max \dot{\theta}_j = 1\frac{1}{s}$ maximal rate of gain-scheduled parameter changes. Constraints for the Lyapunov matrix (10.12) are $0 < P(\xi, \theta, \alpha) < 1000 * I$. The obtained first gain-scheduled controller parameters are:

$$\begin{aligned}
&\sigma = 1 \\
&\quad R_1 = -7.8434 - \frac{3.852}{s} + \left\{-4.794 - \frac{2.6287}{s}\right\}\theta_1 \\
&\qquad + \left\{0.3278 + \frac{0.866}{s}\right\}\theta_2 \\
&\sigma = 2 \\
&\quad R_2 = -13.8691 - \frac{6.3453}{s} + \left\{-8.0619 - \frac{4.7241}{s}\right\}\theta_1 \\
&\qquad + \{0.4297 + \frac{1.2511}{s}\}\theta_2
\end{aligned} \tag{10.26}$$

The maximal closed-loop eigenvalues for the case of $\theta_j = 0$, $j = 1, 2, \ldots, p$ are

$$\sigma = 1, maxeig(CLS) = -0.4899$$
$$\sigma = 2, maxeig(CLS) = -0.4888$$

Under the same conditions other results have been obtained for the case of $\max \dot{\alpha}_\sigma = 300/s$ and $max\, \dot{\theta}_j = 5/s$. The second gain-scheduled controller parameters are:

$$\begin{aligned}
&\sigma = 1 \\
&\quad R_1 = -7.1769 - \frac{2.0309}{s} + \left\{-1.2374 + \frac{2.4355}{s}\right\}\theta_1 \\
&\qquad + \left\{4.9032 - \frac{2.7108}{s}\right\}\theta_2
\end{aligned} \tag{10.27}$$

$$\begin{aligned}
&\sigma = 2 \\
&\quad R_2 = -3.9991 - \frac{1.7337}{s} + \left\{-02912 + \frac{1.3464}{s}\right\}\theta_1 \\
&\qquad + \left\{1.1521 - \frac{1.4535}{s}\right\}\theta_2
\end{aligned} \tag{10.28}$$

The maximal closed-loop eigenvalues for the case of $\theta_j = 0$, $j = 1, 2, \ldots, p$ are

$$\sigma = 1, maxeig(CLS) = -0.2656$$
$$\sigma = 2, maxeig(CLS) = -0.5729$$

Note that the negative sign means a negative feedback. Simulation results for the two designed gain-scheduled switched controller are given in *Figs. 10.1-10.6*

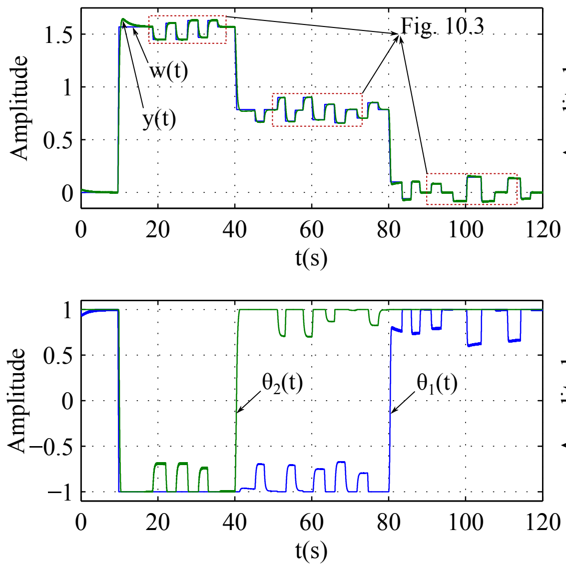

FIGURE 10.1: Simulation results $w(t)$, $y(t)$ with the first gain-scheduled switched controller

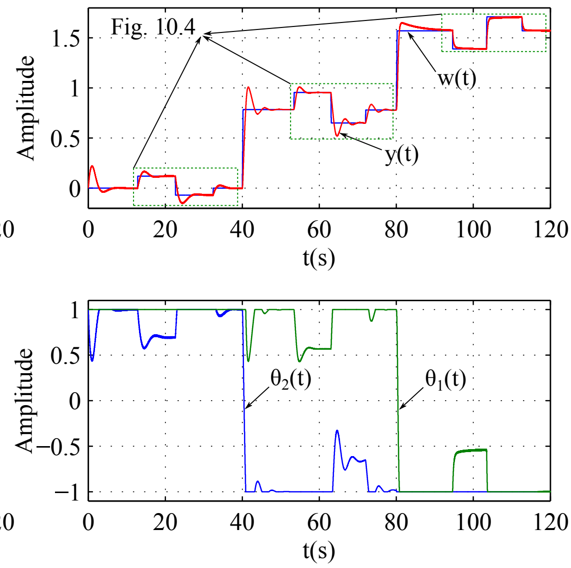

FIGURE 10.2: Simulation results $w(t)$, $y(t)$ with second gain-scheduled switched controller

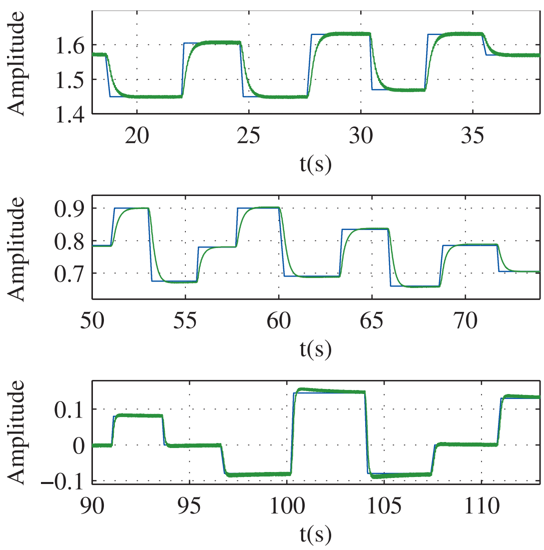

FIGURE 10.3: Simulation results $w(t)$, $y(t)$ with the first gain-scheduled switched controller – zoomed

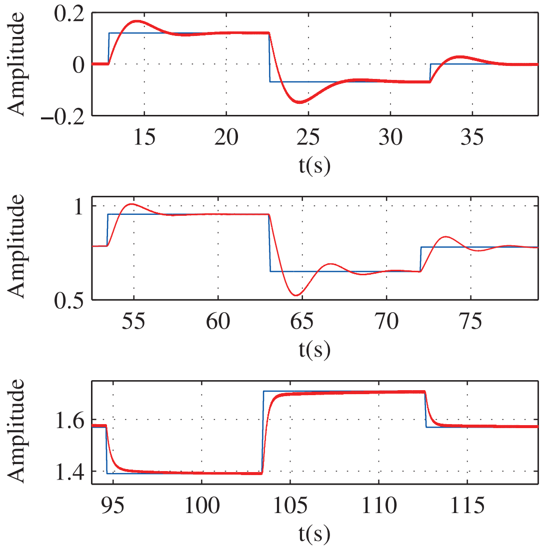

FIGURE 10.4: Simulation results $w(t)$, $y(t)$ with second gain-scheduled switched controller – zoomed

10.5 Conclusion

In the paper a novel switched robust gain-scheduled controller design procedure has been proposed for switched control of nonlinear systems. The proposed method is based on an uncertain gain-scheduled plant, multi parameter dependent Lyapunov function and guaranteed cost. To access the system performance, we consider an original weighted scheduled quadratic cost function which allowed to obtain different performance dependence on the working points, which opens new possibilities for the controller designer. The

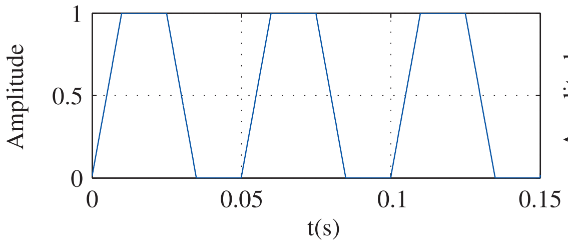

FIGURE 10.5: Development of the switching variable $\alpha_1(t)$

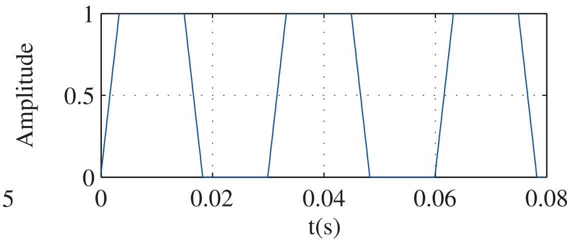

FIGURE 10.6: Development of the switching variable $\alpha_1(t)$

obtained results, illustrated on examples, show the applicability of the designed switched robust gain-scheduled controller and its ability to cope with model uncertainties. In the paper several forms of parameter dependent/quadratic Lyapunov functions are proposed. The obtained results are in the form of BMI. The proposed approach contributes to the design tools for switched robust gain-scheduled controllers for nonlinear systems.

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11

Gain-Scheduled MPC Design for Nonlinear Systems with Input Constraints (Paper 8)

Abstract

A novel methodology is proposed for discrete model predictive gain-scheduled controller design for nonlinear systems with input(hard)/output(soft) constraints for finite and infinite prediction horizons. The proposed design procedure is based on the linear parameter-varying (LPV) paradigm, affine parameter-dependent quadratic stability and on the notion of the parameter-varying guaranteed cost. The obtained design procedure is in the form of BMI. Numerical examples show the benefit of the proposed approach.

Keywords: Gain-scheduled controller, Predictive controller, Parameter-dependent Lyapunov function, Quadratic gain-scheduled cost function, LPV systems, Nonlinear systems.

11.1 Introduction

The robust control theory is well established for linear systems but almost all real processes are more or less nonlinear. If the plant operating region is small, one can use the robust control approaches to design a linear robust controller where the nonlinearities are treated as model uncertainties. However, for real nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable. For this reason the controller design for nonlinear systems is nowadays a very determinative and important field of research.

Gain scheduling is one of the most common used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. Many of the early articles were associated with flight control [1, 2] and aerospace [3]. Then, gradually, this approach has been used almost everywhere in control engineering, which was greatly helped with the introduction of LPV systems. Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. They were introduced first by Jeff S. Shamma in 1988 to model gain scheduling. Today the LPV paradigm has become a standard formalism in systems and controls with lot of researches and articles devoted to analysis, controller design and system identification of these models [4].

It is known that every real system contains some constraints and it is necessary to take account of them in the controller design procedure. The basic solution to this problem was to include some anti-windup techniques in gain scheduling [5, 6].

A more systematic approach which allowed new possibilities is the model predictive control (MPC). That is why MPC attracted a lot of practitioners and became one of the most used advanced control techniques in industrial applications. More information can be found in survey [7]. The underlying idea of MPC is to use the system model to predict the future system behaviour and then to find an optimal system input by minimization of a cost function. Although the MPC was successfully applied to a wide range of industrial processes, it contains some limitations which are caused by the drawbacks in the MPC formulation. In the standard MPC without modifications the closed-loop stability is not guaranteed, not mention robust stability and the computational complexity of QP (quadratic programming) solver in each sample time and the feasibility of the cost function with constraints. For this reason, the MPC and the nonlinear MPC has received much attention in this research area [8, 9]. In many articles the nonlinear system is described by LPV approximation or by gain scheduling. In paper [10] one can find a robust output feedback MPC design for LPV systems where the control law is computed based on LMI at each sampling time. The authors in [11] presented an observer-based controller for nonlinear systems, where the control law is generated using the Jacobian linearization in conjunction with gain scheduling. In paper [12] one can find a stabilizing scheduled output feedback MPC algorithm for constrained nonlinear systems with large operating regions, where the authors design a set of local output feedback predictive controllers with their estimated regions of stability covering the desired operating region, then on-line switches between them with achieving the nonlinear transitions with guaranteed stability. Another MPC algorithm can be found for LPV systems in papers [13] and [14]. Robust output feedback MPC using off-line LMI can be found in [15] and off-line MPC based on gain scheduling for networked control system in a brief paper [16].

The main motivation of our paper were our previous results in gain scheduling [17], [18], [19] and the results from a stable model predictive control design [20]. Following the literature in this paper we have proposed to combine the gain scheduling approaches with the stable MPC design to obtain a new controller design procedure. In this paper a novel static-output model predictive gain-scheduled controller design for finite and infinite horizon is presented for discrete nonlinear systems, which will guarantee the closed-loop stability and performance quality with considering input/output constraints and all this without on-line optimization in each sample time.

The sequel of the manuscript is organized as follows. In *Section 11.2* we present problem formulation and preliminaries. In *Section 11.3* we address the main results which include the model predictive gain-scheduled controller design procedure for nonlinear constrained discrete-time systems for finite and infinite prediction horizon. Finally, in *Section 11.4* the proposed design procedure is demonstrated on simple examples.

Our notations are standard, $D \in \mathbb{R}^{m\times n}$ denotes the set of real $m \times n$ matrices. I_m is an $m\times m$ identity matrix and Z_m denotes a zero matrix. If the size can be determined from the context, we will omit the subscript. $P > 0$ ($P \geq 0$) is a real symmetric, positive definite (semidefinite) matrix.

11.2 Problem formulation and preliminaries

Consider a nonlinear plant $x(k+1) = F(x(k), u(k), \theta(k))$ which is identified in several working points. The identified family of linear systems in discrete-time space is given as follows

$$\begin{aligned} x(k+1) &= A_i\, x(k) + B_i\, u(k) \\ y(k) &= C_i\, x(k) \quad i = 1, 2, \dots N \end{aligned} \tag{11.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the controller output, $y(k) \in \mathbb{R}^l$ is the measured plant output vector at step $k \in \mathbb{R}_+$, matrices A_i, B_i, C_i, $i = 1, 2, \dots, N$ are system matrices with appropriate dimension and N is the number of identified plants model. Assume that a known vector $\theta(k) \in \Omega$ exists which captures the parametric dependence of the linearized model (11.1) on the equilibrium (working) points of the original nonlinear system.

11.2.1 Case of finite prediction horizon

The identified family of linear systems (11.1) for a given prediction and control horizon N_k can be transformed to the following form [20]

$$\begin{aligned} z(k+1) &= A_{f_i}\, z(k) + B_{f_i}\, v(k) \\ y_f(k) &= C_{f_i}\, z(k) \quad i = 1, 2, \dots N \end{aligned} \tag{11.2}$$

where

$$\begin{aligned} z^T(k) &= \left[x^T(k|k)\; x^T(k+1|k) \cdots\; x^T(k+N_k-1|k)\right] \\ v^T(k) &= \left[u^T(k|k)\; u^T(k+1|k) \cdots\; u^T(k+N_k-1|k)\right] \\ y_f^T(k) &= \left[y^T(k|k)\; y^T(k+1|k) \cdots\; y^T(k+N_k-1|k)\right] \end{aligned}$$

$$A_{f\,i} = \begin{bmatrix} A_i & 0 \cdots 0 \\ A_i^2 & 0 \cdots 0 \\ \vdots & \vdots \ddots 0 \\ A_i^{N_k} & 0 \cdots 0 \end{bmatrix}, \quad C_{f\,i} = \begin{bmatrix} C_i & 0 & \cdots & 0 \\ 0 & C_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_i \end{bmatrix}$$

$$B_{f\,i} = \begin{bmatrix} B_i & 0 & \cdots & 0 \\ A_i B_i & B_i & \cdots & 0 \\ A_i^2 B_i & A_i B_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_i^{N_k-1} B_i & A_i^{N_k-2} B_i & \cdots & B_i \end{bmatrix}$$

and $x(k+j|k)$ is the step ahead prediction of the state, calculated in sample time k. From the family of linear systems (11.2) one obtains [17] a gain-scheduled plant model in the form

$$\begin{aligned} z(k+1) &= A_{f\,a}(\theta(k))\, z(k) + B_{f\,a}(\theta(k))\, v(k) \\ y_f(k) &= C_{f\,a}\, z(k) \end{aligned} \tag{11.3}$$

where $C_{f\,a} = C_{f\,1} = C_{f\,2} = \ldots = C_{f\,N}$ and

$$\begin{aligned} A_{f\,a}(\theta(k)) &= A_{f\,a0} + \sum_{i=1}^{p} A_{f\,ai}\theta_i(k) \\ B_{f\,a}(\theta(k)) &= B_{f\,a0} + \sum_{i=1}^{p} B_{f\,ai}\theta_i(k) \end{aligned}$$

and $A_{f\,ai}$, $B_{f\,ai}$, $i = 0, 1, \ldots, p-1$ are system matrices with appropriate dimension, $A_{f\,ap} = 0$, $B_{f\,ap} = 0$, $\theta(k)^T = [\theta_1(k), \theta_2(k), \ldots, \theta_{p-1}(k)] \in \Omega$ is the vector of $p-1$ known independent scheduling parameters at step k and $\theta_p \in \langle 0, H_m \rangle$ is the scheduled parameter which is used to ensure I/O constraints, where $H_m \in (0, 1)$. The control law for the model predictive gain-scheduled controller design for a given prediction and control horizon N_k is considered in the form

$$v(k) = F(\theta(k))\, y_f(k) = F(\theta(k))\, C_{f\,a}\, z_f(k) \tag{11.4}$$

where $F(\theta(k)) = F_0 + \sum_{j=1}^{p-1} F_j \theta_j(k) - F_0 \theta_p(k)$.

Note 11.1. We can extend system (11.3) to PS or PSD control, for more information see [18].

The procedure to ensure the input/output constraints is very simple. If the system input or output approach the maximal or minimal value, using the scheduling parameter θ_p one can affect the controller output. There are several solutions how to generate the scheduling parameter θ_p, it is depending on the system. We will deal with this issue in the examples. If we substitute control law (11.4) to system (11.3), a closed-loop system is obtained

$$z(k+1) = A_c(\theta(k))z(k) \tag{11.5}$$

where $A_c(\theta(k)) = A_{f\,a}(\theta(k)) + B_{f\,a}(\theta(k)) F(\theta(k)) C_{f\,a}$.

To assess the performance quality with possibility to obtain different performance quality in each working point a quadratic cost function described in paper [18] will be used

$$\begin{aligned} J_{df}(\theta(k)) = \sum_{k=0}^{\infty} z_f(k)^T Q(\theta(k)) z_f(k) + v(k)^T R v(k) \\ + \Delta z_f(k)^T S(\theta(k)) \Delta z_f(k) = \sum_{k=0}^{\infty} J_d(\theta(k)) \end{aligned} \quad (11.6)$$

where $\Delta z_f(k) = z_f(k+1) - z_f(k)$, $Q(\theta(k)) = Q_0 + \sum_{i=1}^{p} Q_i\theta_i(k)$, $S(\theta(k)) = S_0 + \sum_{i=1}^{p} S_i\theta_i(k)$, $Q_i = Q_i^T \geq 0$, $S_i = S_i^T \geq 0$, $R > 0$ and $Q_p = S_p = 0$.

Note 11.2. Using the cost function (11.6) we can affect the performance quality separately in each working point with defining different weighting matrices for each working point which then are transformed to affine form and depend on the scheduled parameters as system matrices. [18]

Definition 11.1. Consider system (11.3) with control algorithm (11.4). If a control law v^* and a positive scalar J_d^* exist such that the closed-loop system (11.5) is stable and the value of closed-loop cost function (11.6) satisfies $J_d \leq J_d^*$, then J_d^* is said to be a guaranteed cost and v^* is said to be guaranteed cost control law for system (11.3).

Substituting the control law (11.4) to the quadratic cost function (11.6) one can obtain

$$J_d(\theta(k)) = \tilde{z}^T \begin{bmatrix} J_{d11}(\theta(k)) \; J_{d12}(\theta(k)) \\ J_{d12}^T(\theta(k)) \; J_{d22}(\theta(k)) \end{bmatrix} \tilde{z} \quad (11.7)$$

where $\tilde{z}^T = \begin{bmatrix} z^T(k+1) \; z^T(k) \end{bmatrix}$ and

$$\begin{aligned} J_{d11}(\theta(k)) &= S(\theta(k)), \quad J_{d12}(\theta(k)) = -S(\theta(k)), \\ J_{d22}(\theta(k)) &= Q(\theta(k)) + C_{fa}^T F(\theta(k))^T R\, F(\theta(k))\, C_{fa} \\ &\quad + S(\theta(k)) \end{aligned}$$

To ensure the Affine Quadratic Stability (AQS) [21] the following Lyapunov function has been chosen

$$V(\theta(k)) = z_f^T(k) P(\theta(k)) z_f(k) \quad (11.8)$$

The first difference of Lyapunov function (11.8) is given as follows

$$\begin{aligned} \Delta V(\theta(k)) = z_f^T(k+1)\, P(\theta(k+1))\, z_f(k+1) - \\ - z_f^T(k)\, P(\theta(k))\, z_f(k) \end{aligned} \quad (11.9)$$

where

$$P(\theta(k)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) \quad (11.10)$$

On substituting $\theta(k+1) = \theta(k) + \Delta\theta(k)$ to $P(\theta(k+1))$ one obtains the following result

$$P(\theta(k+1)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + \sum_{i=1}^{p} P_i\Delta\theta_i(k) \tag{11.11}$$

where if assuming that $P_i > 0$, $\Delta\theta_i \in \langle \Delta\underline{\theta_i}, \Delta\overline{\theta_i}\rangle \in \Omega_t$, $i = 0, 1, \ldots p$ and $\max|\Delta\theta_i| < \rho_i$, one can write

$$P(\theta(k+1)) \leq P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + P_\rho = P_\rho(\theta(k)) \tag{11.12}$$

where $P_\rho = \sum_{i=1}^{p} P_i\rho_i$. The first difference of the Lyapunov function (11.9) using the free matrix weighting approach [17] is in the form

$$\Delta V(\theta(k)) = \tilde{z}^T \begin{bmatrix} V_{11}(\theta(k)) & V_{12}(\theta(k)) \\ V_{12}^T(\theta(k)) & V_{22}(\theta(k)) \end{bmatrix} \tilde{z} \tag{11.13}$$

where

$$\begin{aligned} V_{11}(\theta(k)) &= P_\rho(\theta(k)) + N_1 + N_1^T \\ V_{12}(\theta(k)) &= N_2^T - N_1 A_c(\theta(k)) \\ V_{22}(\theta(k)) &= -P(\theta(k)) - N_2 A_c(\theta(k)) - A_c^T(\theta(k)) N_2^T \end{aligned}$$

where $N_1, N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices.

Definition 11.2. [21] The linear closed-loop system (11.5) for $\theta(k) \in \Omega$ and $\Delta\theta(k) \in \Omega_t$ is affinely quadratically stable if and only if $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ exist such that $P(\theta(k))$ (11.10), $P_\rho(\theta(k))$ (11.12) are positive defined and for the first difference of the Lyapunov function (11.13) along the trajectory of closed-loop system (11.5) it holds

$$\Delta V(\theta(k)) < 0 \tag{11.14}$$

From LQ theory we can introduce the well known results:

Lemma 11.1. *Consider the closed-loop system (11.5). Closed-loop system (11.5) is affinely quadratically stable with guaranteed cost if and only if the following inequality holds*

$$B_e(\theta(k)) = \min_u \{\Delta V(\theta(k)) + J_d(\theta(k))\} \leq 0 \tag{11.15}$$

for all $\theta(k) \in \Omega$. For proof see [22].

11.2.2 Case of infinite prediction horizon

The system described by (11.3) for the case of $N_k = 0$ can be transformed to the gain-scheduled plant model

$$\begin{aligned} x(k+1) &= A(\theta(k))\,x(k) + B(\theta(k))\,u(k) \\ y(k) &= C\,x(k) \end{aligned} \tag{11.16}$$

where $A(\theta(k)) = A_0 + \sum_{i=1}^{p} A_i\theta_i(k)$, $B(\theta(k)) = B_0 + \sum_{i=1}^{p} B_i\theta_i(k)$, $A_p = 0$ and $B_p = 0$. For the case $N_k \to \infty$ and $S = 0$ the cost function (11.6) can be rewritten as

$$J = \sum_{k=0}^{\infty} J(k) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} x^T(k+j) q_j x(k+j) \right. \\ \left. + u^T(k+j) r_j u(k+j) \right) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \tilde{J}(k) \tag{11.17}$$

where $q_j \in \mathbb{R}^{n\times n}$, $r_j \in \mathbb{R}^{m\times m}$ are positive definite matrices. The control law for the model predictive gain-scheduled controller design for the infinite prediction horizon is considered in the form

$$u(k) = F(\theta(k))\, y(k) = F(\theta(k))\, C\, x(k) \tag{11.18}$$

where $F(\theta(k)) = F_0 + \sum_{j=1}^{p-1} F_j\theta_j(k) - F_0\theta_p(k)$. To guarantee the stability and performance of the closed-loop gain-scheduled system, due to *Lemma 11.1* it is sufficient to ensure

$$B_e(\theta(k)) = \Delta V(x(k+j), \theta(k)) + \tilde{J}(k) \leq 0 \tag{11.19}$$

where $\Delta V(x(k+j), \theta(k)) = V(x(k+j+1), \theta(k) + \Delta\theta(k)) - V(x(k), \theta(k))$ is the first difference of the Lyapunov function for j horizon prediction. Summing (11.19) from $j = 0$ to $j \to \infty$, the upper bound on $J(k)$ is obtained

$$J(k) \leq V(x(k), \theta(k)) \tag{11.20}$$

On the basis of (11.20) the following gain-scheduled MPC design procedure is given

$$\min_{F(\theta(k))} V(x(k), \theta(k)) \tag{11.21}$$

with constraints to system model (11.16), stability model and performance (11.19) and other constraints. Assume that the Lyapunov function is in the form $V(x(k), \theta(k)) = x^T(k)P(\theta(k))x(k)$, where $P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k)$. Due to (11.21), the predicted control design procedure can be modified as

$$\min_{F(\theta(k))} x^T(k)P(\theta(k))x(k) \leq x^T(k)x(k)\gamma \tag{11.22}$$

which leads to the inequality

$$P(\theta(k)) \leq \min_{F(\theta(k))} \gamma I \tag{11.23}$$

In the paper [23] inequality (11.22) is in the form

$$\min x^T(k)P(\theta(k))x(k) \leq \gamma \tag{11.24}$$

which needs to know the state vector $x(k)$ and on-line optimization of (11.24) at every sample time.

The stability of the closed-loop system is guaranteed if

$$B_e = \Delta V(x(k), \theta(k)) + \alpha V(x(k), \theta(k)) \leq 0 \tag{11.25}$$

where $\alpha \in \langle 0, 1)$ is a coefficient with an influence on the closed-loop system performance. If we substitute the Lyapunov function and its first difference to (11.25), we can obtain

$$B_e(\theta(k)) = \tilde{x}^T W(\theta(k))\, \tilde{x} \leq 0 \tag{11.26}$$

where $\tilde{x}^T = \left[x^T(k+1)\, x^T(k)\right]$, $\beta = 1 - \alpha$ and

$$\begin{aligned} W(\theta(k)) &= \begin{bmatrix} W_{11}(\theta(k)) & W_{12}(\theta(k)) \\ W_{12}(\theta(k))^T & W_{22}(\theta(k)) \end{bmatrix} \\ W_{11}(\theta(k)) &= N_1^T + N_1 + P_\rho(\theta(k)) \\ W_{12}(\theta(k)) &= -N_1^T A_c(\theta(k)) + N_2 \\ W_{22}(\theta(k)) &= -N_2 A_c(\theta(k)) - A_c^T(\theta(k)) N_2 - P(\theta(k))(\beta) \end{aligned}$$

11.3 Main results

In this section the discrete predictive gain-scheduled controller design procedure is presented which guarantees the affine quadratic stability and guaranteed cost for $\theta(k) \in \Omega$ with pre-defined maximal rate of change of the scheduled parameters ρ. The main result of this section – the discrete model predictive gain-scheduled controller design procedure – relies on the concept of multi-convexity, that is convexity along each direction $\theta_i(k),\ i = 1, 2, \ldots, p$ of the parameter space. The implications of multiconvexity for scalar quadratic functions are given in the next lemma [21].

Lemma 11.2. *Consider a scalar quadratic function of $\alpha \in \mathbb{R}^p$.*

$$f(\alpha) = a_0 + \sum_{i=1}^{p} a_i \alpha_i + \sum_{i=1}^{p} \sum_{j>i}^{p} b_{ij} \alpha_i \alpha_j + \sum_{i=1}^{p} c_i \alpha_i^2 \tag{11.27}$$

and assume that $f(\alpha_1, \ldots, \alpha_p)$ is multi-convex, that is $\frac{\partial^2 f(\alpha)}{\partial \alpha_i^2} = 2c_i \geq 0$ for $i = 1, 2, \ldots, p$. Then $f(\alpha)$ is negative for all $\alpha \in \Omega$ and $\dot{\alpha} \in \Omega_t$ if and only if it takes negative values at the corners of α.

11.3.1 Finite prediction horizon

Using *Lemmas 11.1* and *11.2* the following theorem is obtained for discrete model predictive gain-scheduled controller design for finite horizon.

Theorem 11.1. *Closed-loop system (11.5) is affinely quadratically stable if $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ exist such that $P(\theta(k))$ (11.10), $P_\rho(\theta(k))$ (11.12) are positive definite for all $\theta(k) \in \Omega$, with pre-defined ρ_i, matrices N_1, N_2, Q_i, R, S_i, $i = 1, 2, \ldots p$*

and gain-scheduled matrices $F(\theta(k))$ satisfying

$$\begin{aligned} M(\theta(k)) < 0; &\quad \theta(k) \in \Omega \\ M_{ii} \geq 0; &\quad i = 1, 2, \ldots, p \end{aligned} \tag{11.28}$$

where (at sample time k)

$$\begin{aligned}
M(\theta) &= M_0 + \textstyle\sum_{i=1}^{p} M_i\theta_i + \sum_{i=1}^{p}\sum_{j>i}^{p-1} M_{ij}\theta_i\theta_j + \sum_{i=1}^{p} M_{ii}\theta_i^2 \\
M_0 &= \begin{bmatrix} M_{110} & M_{120} \\ M_{120}^T & M_{220} \end{bmatrix}, \; M_i = \begin{bmatrix} M_{11i} & M_{12i} \\ M_{12i}^T & M_{22i} \end{bmatrix} \\
M_{ij} &= \begin{bmatrix} M_{11ij} & M_{12ij} \\ M_{12ij}^T & M_{22ij} \end{bmatrix}, M_{110} = P_0 + N_1 + N_1^T + S_0 + P_\rho \\
M_{11i} &= P_i, \; M_{11ij} = 0, \; M_{11ii} = 0 \\
M_{120} &= N_2 - N_1^T(A_{fa0} + B_{fa0}F_0C_{fa}) - S_0 \\
M_{12i} &= -N_1^T(A_{fai} + B_{fai}F_0C_{fa} + B_{fa0}F_iC_{fa}) - S_i \\
M_{12ij} &= -N_1^T(B_{fai}F_j + B_{faj}F_i)C_{fa} \\
M_{12ii} &= -N_1^T B_{fai}F_iC_{fa} \\
M_{220} &= Q_0 + S_0 - P_0 - N_2^T(A_{fa0} + B_{fa0}F_0C_{fa}) \\
&\quad -(A_{fa0} + B_{fa0}F_0C_{fa})^T N_2 + C_{fa}^T F_0^T RF_0C_{fa} \\
M_{22i} &= -P_i - N_2^T(A_{fai} + B_{fai}F_0C_{fa} + B_{fa0}F_iC_{fa}) \\
&\quad -(A_{fai} + B_{fai}F_0C_{fa} + B_{fa0}F_iC_{fa})^T N_2 \\
&\quad +C_{fa}^T(F_0^T RF_i + F_i^T RF_0)C_{fa} + Q_i + S_i \\
M_{22ij} &= -N_2^T(B_{fai}F_j + B_{faj}F_i)C_{fa} - C_{fa}^T(B_{fai}F_j \\
&\quad +B_{faj}F_i)^T N_2 + C_{fa}^T(F_i^T RF_j + F_j^T RF_i)C_{fa} \\
M_{22ii} &= -N_2^T B_{fai}F_iC_{fa} - (B_{fai}F_iC_{fa})^T N_2 \\
&\quad +C_{fa}^T F_i^T RF_iC_{fa}
\end{aligned}$$

Proof. The proof of the *Theorem 11.1* is clear from the previous derivations. Here, the proof is repeated only in basic steps. The proof is based on the *Lemmas 11.1* and *11.2*. When substituting the first difference of the Lyapunov function (11.13) and the quadratic cost function (11.7) to the Bellman-Lyapunov function (11.15), after some manipulation, using *Lemma 11.2* we obtain (11.28) which proofs the *Theorem 11.1*. □

11.3.2 Infinite prediction horizon

Using inequalities (11.23), (11.26) and *Lemma 11.2* the following theorem is obtained for discrete model predictive gain-scheduled controller design for infinite horizon.

Theorem 11.2. *Closed-loop system is affinely quadratically stable if there exist $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that $P(\theta(k))$ (11.10), $P_\rho(\theta(k))$ (11.12) are positive definite for all $\theta(k) \in \Omega$, with pre-defined ρ_i, matrices N_1, N_2, and gain-scheduled matrices $F(\theta(k))$ satisfying*

$$\begin{aligned} W(\theta(k)) < 0; &\quad \theta(k) \in \Omega \\ W_{ii} \geq 0; &\quad i = 1, 2, \ldots, p \\ P(\theta(k)) \leq &\min_{F(\theta(k))} \gamma \end{aligned} \tag{11.29}$$

where (at sample time k)

$$
\begin{aligned}
W(\theta) &= W_0 + \textstyle\sum_{i=1}^{p} W_i\theta_i + \sum_{i=1}^{p}\sum_{j>i}^{p-1} W_{ij}\theta_i\theta_j + \sum_{i=1}^{p} W_{ii}\theta_i^2 \\
W_0 &= \begin{bmatrix} W_{110} & W_{120} \\ W_{120}^T & W_{220} \end{bmatrix}, \quad W_i = \begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix} \\
W_{ij} &= \begin{bmatrix} W_{11ij} & W_{12ij} \\ W_{12ij}^T & W_{22ij} \end{bmatrix}, \quad W_{110} = P_0 + N_1 + N_1^T + P_\rho \\
W_{11i} &= P_i, \; W_{11ij} = 0, \; W_{11ii} = 0 \\
W_{120} &= N_2 - N_1^T(A_0 + B_0F_0C) \\
W_{12i} &= -N_1^T(A_i + B_iF_0C + B_0F_iC), \\
W_{12ij} &= -N_1^T(B_iF_j + B_jF_i)C \\
W_{220} &= -N_2^T(A_0 + B_0F_0C) - (A_0 + B_0F_0C)^TN_2 - P_0\beta \\
W_{22i} &= -N_2^T(A_i + B_iF_0C + B_0F_iC) \\
&\quad -(A_i + B_iF_0C + B_0F_iC)^TN_2 - P_i\beta \\
W_{22ij} &= -N_2^T(B_iF_j + B_jF_i)C - C^T(B_iF_j + B_jF_i)^TN_2 \\
W_{12ii} &= -N_1^TB_iF_iC, \; W_{22ii} = -N_2^TB_iF_iC - (B_iF_iC)^TN_2
\end{aligned}
$$

Proof. The proof of the *Theorem 11.2* regarding to space limitations is sketched only in basic steps. The proof is based on the *Lemmas 11.1* and *11.2*. If we substitute the Lyapunov function and its first difference to (11.25), we can obtain (11.26), after some manipulation, using *Lemma 11.2* we obtain (11.29) which proofs the *Theorem 11.2*. □

11.4 Examples

The first example will be a simple nonlinear system which has an unstable zero equilibrium point. The system is borrowed from [11]:

$$
\begin{aligned}
\dot{x} &= -x|x| + u \qquad -0.5 \le u \le 0.5 \\
y &= x \qquad\qquad\quad\; -0.5 \le y \le 0.5
\end{aligned} \tag{11.30}
$$

The system (11.30) is transformed into the following form

$$
\begin{aligned}
\dot{x} &= -a(\theta)x + bu \qquad -0.5 \le u \le 0.5 \\
y &= cx \qquad\qquad\qquad -0.5 \le y \le 0.5
\end{aligned} \tag{11.31}
$$

where $a(\theta) = a_0 + a_1\theta$, $b = 1$, $c = 1$ and $\theta = \frac{y|y| - a_0 y}{a_1 y} \in \langle -1, 1\rangle$. The coefficients a_0 and a_1 were calculated so as to maintain the scheduling parameter θ in the range $\langle -1, 1\rangle$

$$
h = \max\left(\frac{y|y|}{y}\right); l = \min\left(\frac{y|y|}{y}\right); a_0 = \frac{l+h}{2}; a_1 = \frac{l-h}{2}
$$

In our case for $y \in \langle -0.5, 0.5\rangle$ $h = 0.5$, $l = 0$ and it follows that $a_0 = 0.25$ and $a_1 = -0.25$. If one substitutes the extreme points of $\theta \in \langle -1, 1\rangle$ to the obtained LPV system (11.31) one obtains 2 linear systems. These linear systems were transformed to the discrete time-space with sample time $T_{st} = 0.01\,s$ and then extended for model predictive controller design with control and prediction horizon $N_k = 4$ to obtain model in the form (11.2). Then the obtained systems were transformed to the form (11.3) with scheduling parameters $\theta_1 \in \langle -1, 1\rangle$ and $\theta_p = \theta_2 \in \langle 0, H_m\rangle$, $H_m = 0.89$. To achieve the setpoint with zero control deviation we extend the system for PI model predictive controller design.

Using *Theorem 11.1* with weighting matrices $Q = q_i I$, $q_0 = 1 \times 10^{-5}$, $q_1 = q_2 = 0$, $R = rI$, $r = 10$, $S = s_i I$, $s_0 = 1 \times 10^{-50}$, $s_1 = s_2 = 0$ and $\xi_U \leq P(\theta) \leq \xi_L$, $\xi_U = 1 \times 10^4$, $\xi_L = 1 \times 10^{-5}$, $T_{st} = 0.01\,s$ and $\rho_1 = \rho_2 = 10\ s^{-1}$ we obtained discrete model predictive controller in the form (11.4) where

$$F_0 = \begin{bmatrix} -13.0132 & 1.7809 & 0.4441 & -0.4923 & -0.3557 & -0.0963 & 0.3846 & -0.2068 \\ 0.6242 & -0.0185 & -0.4039 & 0.2242 & 0.1684 & -0.1677 & -0.0675 & 0.0605 \\ -0.9858 & 0.9487 & -0.6442 & 0.1888 & 0.0962 & 0.2371 & -0.5820 & 0.2699 \\ -2.0227 & 1.0136 & 0.6677 & -0.4935 & -0.1344 & 0.0092 & 0.3362 & -0.1747 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0.0001 & 0.0002 & -0.0009 & -0.0000 & -0.0002 & -0.0005 & 0.0015 & -0.0007 \\ -0.0075 & -0.0113 & 0.0467 & 0.0009 & 0.0079 & 0.0273 & -0.0770 & 0.0357 \\ -0.0032 & -0.0048 & 0.0197 & 0.0004 & 0.0033 & 0.0115 & -0.0325 & 0.0151 \\ -0.0027 & -0.0040 & 0.0165 & 0.0003 & 0.0028 & 0.0097 & -0.0272 & 0.0126 \end{bmatrix}$$

and $F_2 = -F_0$. To achieve the input constraints we use the following equation to generate parameter θ_2:

$$\theta_2 = \begin{cases} 0 & \rightarrow \text{if } |u| < l_s \\ |u|\left(\frac{-H_m}{l_s - u_s}\right) + \left(H_m - u_s\left(\frac{-H_m}{l_s - u_s}\right)\right) & \rightarrow \text{if } |u| \geq l_s \end{cases}$$

where $u_s = 0.5$ and $l_s = u_s - 0.001$.

Simulation results (*Figs. 11.1, 11.2* and *11.3*) confirm that *Theorem 11.1* holds and system (11.30) is stable with considering the input constraints.

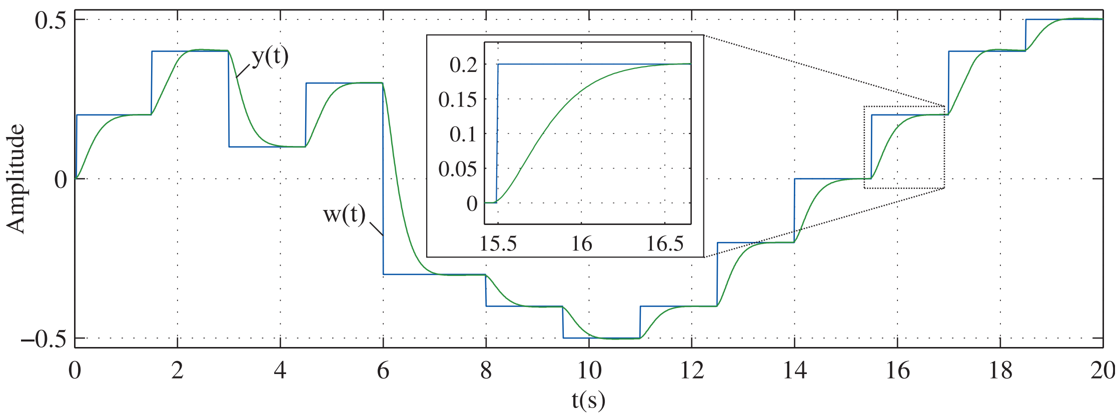

FIGURE 11.1: System output $y(t)$ and the setpoint $w(t)$

The second example is a model of a synchronous generator described by equations as follows

$$\ddot{\delta}T_j + \dot{\delta}D = P_t - P_m \sin(\delta); \dot{P}_t T_m + P_t = ku; P_e = P_m \sin(\delta)$$

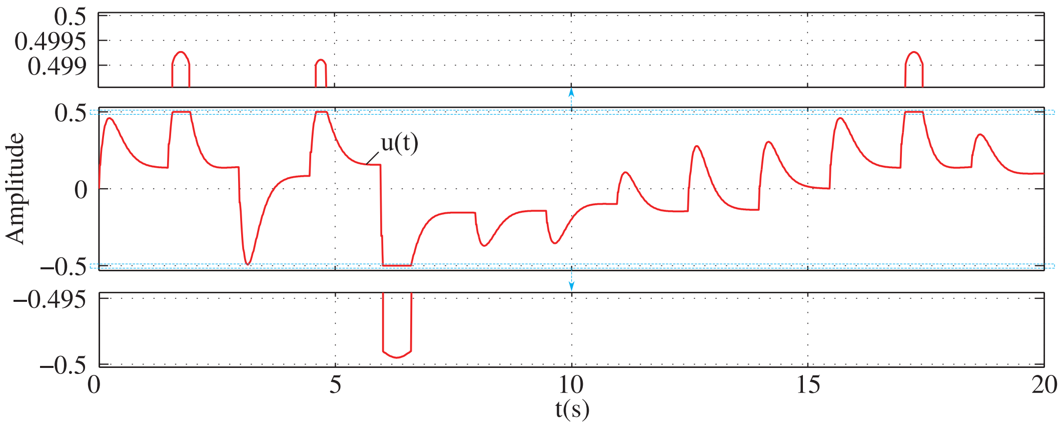

FIGURE 11.2: System input, $u(t)$

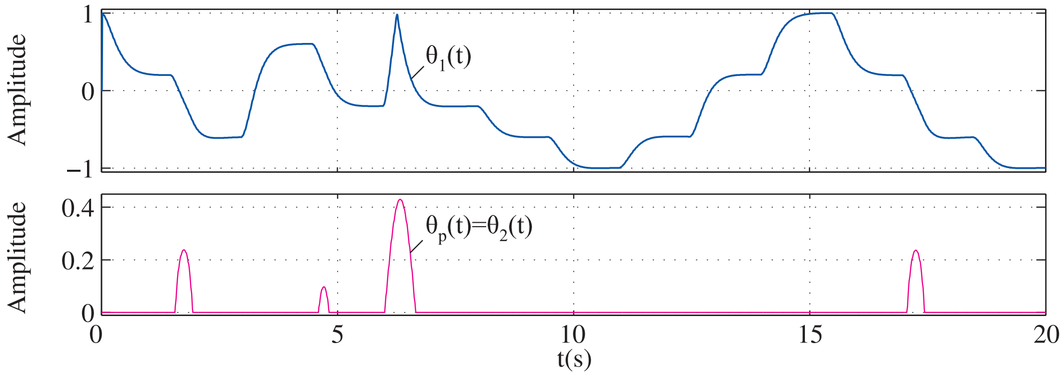

FIGURE 11.3: Calculated scheduling parameters, $\theta(t)$

If we designate $\delta = x_1$, $\dot{\delta} = x_2$, $P_t = x_3$, $P_e = y$ and we select a working trajectory (working points) $\delta = 30,\ 47.5,\ 60$ we can obtain three linear models

$$\begin{array}{l} \dot{x}_1 = x_2; \quad \dot{x}_2 = x_1\left(-\frac{a_i P_m}{T_j}\right) + x_2\left(-\frac{D}{T_j}\right) + x_3\left(\frac{1}{T_j}\right) \\ \dot{x}_3 = x_3\left(-\frac{1}{T_m}\right) + u\left(\frac{1}{T_m}\right); \quad y = x_1\left(a_i P_m\right) \end{array}$$

where $a_i = \frac{\sin(\delta_{wp_i})}{\delta_{wp_i}}$, $i = 1, 2, 3$ and $\delta_{wp_1} = 30$, $\delta_{wp_1} = 47.5$, $\delta_{wp_1} = 60$, $P_m = 1.1\,MVA$, $T_j = 6.58/314\,s^2$, $T_m = 1.5\,s$, $D = 0.01\,s$ and $k = 1$. These three models we transform to discrete time-space with sample time $T_{st} = 0.01\,s$. The obtained model was transformed to the form (11.16) with scheduling parameters $\theta_1,\ \theta_2 \in \langle -1, 1\rangle$ and $\theta_p = \theta_3 \in \langle 0, H_m\rangle$, $H_m = 0.3$. To achieve the setpoint with zero control deviation we extend the system for PI model predictive controller design. Then using *Theorem 11.2* with the coefficient $\alpha = 0.25$ and $\xi_U \leq P(\theta) \leq \xi_L$, $\xi_U = 1 \times 10^8$, $\xi_L = 1 \times 10^{-5}$ and $\rho_1 = \rho_2 = \rho_3 = 10\,s^{-1}$ we obtained the discrete model predictive controller in the form (11.4) where

$$\begin{array}{l} F_0 = \begin{bmatrix} -1.7123 & -0.0150 \end{bmatrix},\ F_1 = \begin{bmatrix} -0.0202 & -0.0002 \end{bmatrix} \\ F_2 = \begin{bmatrix} -0.0185 & -0.0001 \end{bmatrix},\ F_3 = -F_0 \end{array}$$

Simulation results are shown in *Figs. 11.4*, *11.5* and *11.6*.

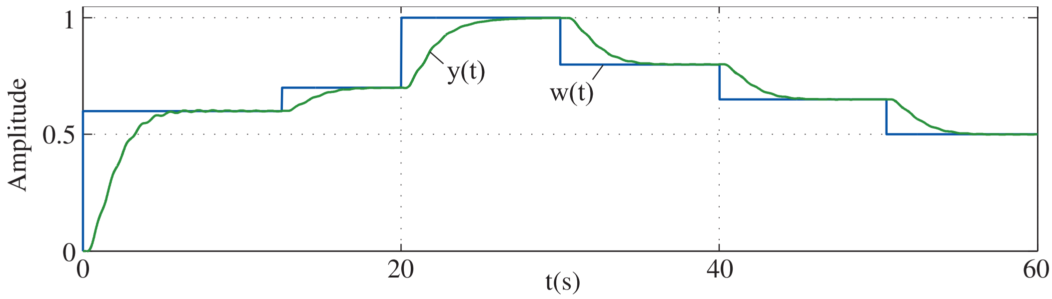

FIGURE 11.4: Simulation results, $y(t), w(t)$

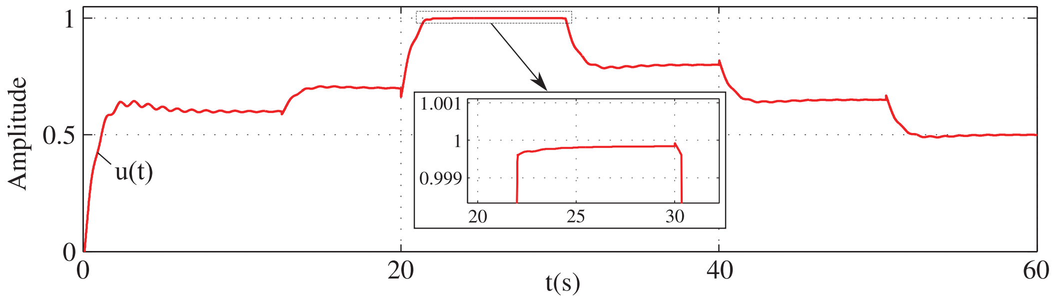

FIGURE 11.5: System input, $u(t)$

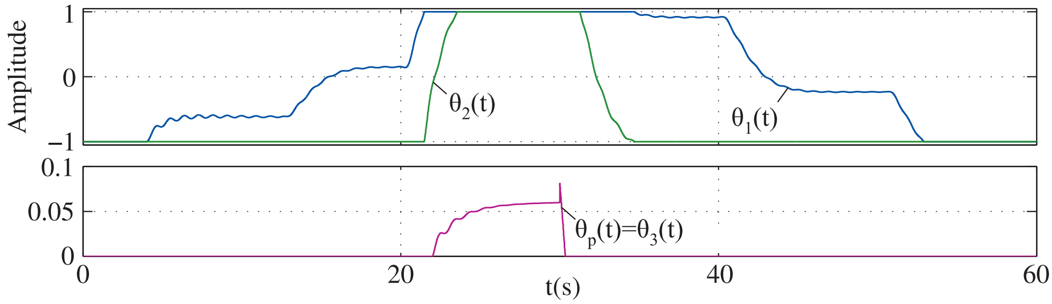

FIGURE 11.6: Calculated scheduling parameters, $\theta(t)$

11.5 Conclusion

In this paper a novel gain scheduling based MPC design procedure is presented for nonlinear systems with I/O constraints for finite and infinite prediction horizons. The design procedure is in the form of BMI (we can use a free and open source BMI solver). Numerical examples show the benefits for the finite and infinite prediction horizon. The presented theory opens new possibilities for further research and study in this area.

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12

Unified Robust Gain-Scheduled and Switched Controller Design for Linear Continuous-Time Systems (Paper 9)

Abstract

In this paper we study the problem how to obtain a new unified procedure to design a robust gain-scheduled and switched controller for continuous-time systems described by a novel robust plant model using the parameter dependent quadratic stability (PDQS) approach. In the proposed design procedure with output feedback a novel quadratic cost function is proposed which allows to obtain different performance dependence on the working points. Finally a numerical example is investigated.

Keywords: Gain-scheduled controller, switched system, robust controller, output feedback, parameter dependent quadratic stability.

12.1 Introduction

The topic of robust hybrid systems has attracted considerable attention in the past decades. Wherever continuous and discrete dynamics interact, a hybrid system arises. The main motivation for studying hybrid systems comes from the two facts:

- hybrid systems have numerous applications in the real world, and
- in real control, there are dynamical systems that cannot be stabilized by any continuous static (dynamic) output state controller but a stabilizing hybrid control scheme can be found.

There are several approaches to model hybrid systems [1], [2]. In the references they consider a discrete even system and continuous dynamics modeled by differential or difference equations. Such models are used to formulate general stability conditions for hybrid systems. In this paper, we consider the class of hybrid systems known as switched systems [3]. There are at last two approaches for stability analysis and controller synthesis of switched systems. The quadratic stability approach with the common Lyapunov function gives stability of closed-loop switched systems under an arbitrary switching law and a multiple Lyapunov function which is less conservative. The survey of the present status of switched systems can be consulted in the excellent paper and work of [4], [1] and [5]. A huge number of references can be found in the switched control of linear discrete-time invariant systems but in the field of linear continuous-time invariant systems the number of references is rather small. The representatives are the following references [6], [7], [8], [9], [10].

In real applications a controller must accommodate a plant with changing dynamics. Therefore, controllers based on these models have to be robust in the presence of plant model uncertainty. In a practical situation one could use the gain scheduling approach which involves scheduling in a family of local controllers in response to the changing plant dynamics. The proposed family of local controllers is implemented using the gain scheduling approach. The classical gain scheduling design procedure typically involves the following steps [11]

- The equilibrium operating points are parameterized by an appropriate quantity $\theta^T = [\theta_1, \ldots, \theta_p]$ which may be involve to the plant input and output.
- The plant dynamics is approximated, locally to a specific equilibrium operating point at which one obtains different values of θ.
- For a given controller structure with a value of θ a linear time-invariant controller is designed. It should be noted that θ is assumed to be constant when designing this controller.
- Repeat steps 2 and 3 for a family of operating points. Family of controllers are parameterized by θ.

A survey of gain-scheduled controller design can be found in [11], [12] and [13] using linear controller design techniques. In our paper, using a new uncertain plant model, a unified approach to design a robust gain-scheduled and robust switched controller with an arbitrarily switching law [1] is developed for continuous-time linear systems. Using multiple parameter dependent Lyapunov function the novel robust gain-scheduled and switched controller design procedure is obtained in the form of BMI. In the proposed design procedure there is no need to use the approach of "dwell time" [6], [7] for switched controller design with arbitrary switching. The "dwell time" markedly complicates the robust switched controller design procedure and makes the obtained results conservative. The advantages of the proposed method are as follows:

- a unified approach to robust gain-scheduled and robust switched controller design with arbitrary switching for linear continuous-time system has been developed,
- in the method proposed in the paper for the case of switched controller design with arbitrary switching there is no need to use the approach of "dwell-time",
- the switched controller designer can take into account the non-ideal switching, that is switching a variable with a rate of switching less than infinite, which open the new possibility for the controller designer,
- different performance could be prescribed by the proposed new cost function for all plant modes.

Organization of the paper is following. *Section 12.2* includes problem formulation of the robust gain-scheduled or switched controller design and some preliminaries are given. In *Section 12.3* sufficient stability conditions to design the robust gain-scheduled or switched controller in the form of BMI are given. In *Section 12.4* the obtained results are illustrated on real examples.

12.2 Problem formulation and preliminaries

The class of LPV uncertain continuous-time systems considered in the paper can be represented by the following model

$$\begin{aligned} \dot{x}(t) &= A(\xi,\theta)x(t) + B(\xi,\theta)u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{12.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, $y(t) \in \mathbb{R}^l$ is the output vector of the system to be controlled. Time varying scheduling parameter vector $\theta^T = [\theta_1, \ldots, \theta_p]$ is assumed to belong to a hyper rectangle:

- Gain-Scheduled Controller

$$\theta_i \in \langle \underline{\theta}_i,\, \overline{\theta}_i \rangle,\; i = 1, 2, \ldots, p,\; \underline{\theta}_i = -\overline{\theta}_i,\; \dot{\theta}_i \in \langle \underline{\dot{\theta}_i},\, \overline{\dot{\theta}_i} \rangle \tag{12.2}$$

- Switched Controller Design

$$\theta_i \in \langle 0,\, 1 \rangle,\; i = 1, 2, \ldots, p,\; \sum_{i=1}^{p} \theta_i = 1,\; \sum_{i=1}^{p} \dot{\theta}_i = 0 \tag{12.3}$$

$$\dot{\theta}_i \in \langle \underline{\dot{\theta}_i},\, \overline{\dot{\theta}_i} \rangle$$

where θ_i, $i = 1, 2, \ldots, p$ is a switching variable with an arbitrary switching algorithm and p is the number of switching plant modes. Observation of (12.1) implies that for an arbitrary switching algorithm the number of active switching plant modes generates the value of switching variable θ_i, $i = 1, 2, \ldots, p$ to determine which controller will be active.

- Assume that for both cases hold $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$

There are two possibilities for switching parameters θ_i, $i = 1, 2, \ldots, p$

1. the rates of change of the switching parameters are infinite, ideal switching case, in this case the quadratic stability with respect to parameter θ can be used or
2. the rates of change of switching parameters θ_i are finite, non-ideal case, in this case the PDQS is recommended to use.

Affine dependence on the scheduling (switching) parameter vector is assumed as follows

$$\begin{aligned} A(\xi,\theta) &= A_0(\xi) + \sum_{i=1}^{p} A_i(\xi)\theta_i \quad \in \mathbb{R}^{n\times n} \\ B(\xi,\theta) &= B_0(\xi) + \sum_{i=1}^{p} B_i(\xi)\theta_i \quad \in \mathbb{R}^{n\times m} \end{aligned} \tag{12.4}$$

where $A_i(\xi)$, $B_i(\xi)$, $i = 0, 1, \ldots, p$ belong to a convex polytope box with N vertices that can be formally defined as

$$\begin{aligned} \Omega = &\Bigg\{ A_i(\xi), B_i(\xi) = \sum_{j=1}^{N} (A_{ij}, B_{ij})\xi_j,\ i = 0, 1, \ldots, p, \\ &\sum_{j=1}^{N} \xi_j = 1,\ \xi_j \geq 0,\ \xi_j \in \Omega_\xi \Bigg\} \end{aligned} \tag{12.5}$$

Assume that the entries of vector ξ are unknown but constant. The following Gain-Scheduled (Switched with an arbitrary switching algorithm) dynamic output feedback controller is considered:

$$\begin{aligned} \dot{x}_c &= A_c(\theta)x_c + B_c(\theta)y(t) \\ u &= C_c x_c + D_c y(t) \end{aligned} \tag{12.6}$$

or

$$u = [D_c C \quad C_c][x \quad x_c]^T = F_c[x \quad x_c]^T$$

where $x_c \in \mathbb{R}^{n_c}$, $n_c = n$ represents the controller state vector. Matrices $A_c(\theta)$, $B_c(\theta)$ are supposed to have the following structure

$$\begin{aligned} A_c(\theta) &= A_{c0} + \sum_{i=1}^{p} A_{ci}\theta_i \\ B_c(\theta) &= B_{c0} + \sum_{i=1}^{p} B_{ci}\theta_i \end{aligned} \tag{12.7}$$

Closed-loop system of (12.1) and (12.6) can be written as

$$\dot{z} = \begin{bmatrix} A(\xi,\theta) + B(\xi,\theta)D_c C & B(\xi,\theta)C_c \\ B_c(\theta)C & A_c(\theta) \end{bmatrix} z = A_{cl}(\theta,\xi)z \tag{12.8}$$

where $z^T = [x^T \quad x_c^T]$. Closed-loop system (12.8) is affine to both the uncertain vector parameter ξ and scheduling (switching) vector parameter θ. Let the following cost function be associated with the closed-loop system

$$J = \int_0^\infty \left(z^T Q(\theta) z + u^T R u\right) dt = \int_0^\infty J(t) dt \tag{12.9}$$

where $Q(\theta) \in \mathbb{R}^{z \times z}$ is positive definite (semidefinite) matrix with structure $Q(\theta) = Q_0 + \sum_{i=1}^{p} Q_i \theta_i$ and $R \in \mathbb{R}^{m \times m} > 0$.

Definition 12.1. Consider system (12.1) and controller (12.6). If there exists a control law u^* and a positive scalar J^* such that the respective closed-loop system (12.8) is stable and the value of the closed-loop cost function (12.9) satisfies $J_c \leq J^*$, then J^* is said to be the guaranteed cost and u^* is said to be the guaranteed cost control law for system (12.8).

Lemma 12.1. *[14][15] Consider system* (12.1) *with control algorithm* (12.6). *Control algorithm* (12.6) *is the guaranteed cost control law for the closed-loop system* (12.8) *if and only if there exists a Lyapunov function $V(z, \theta, \xi)$ such that the following condition holds*

$$B_e(z, \theta, \xi) = \min_u \left\{ \frac{dV(z, \theta, \xi)}{dt} + J(t) \right\} \leq -\epsilon z^T z \quad \epsilon \to 0 \tag{12.10}$$

for all $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$.

Uncertain system (12.1) with control algorithm (12.4) conforming to *Lemma 12.1* is called to be robust stable with guaranteed cost. Note that for a concrete structure of $V(z, \theta, \xi)$ *"if and only if"* may to be decreased to *"if"*.

12.3 Robust gain-scheduled and switched controller design

In this Section the unified robust gain-scheduled and switched controller design procedure is presented. In the references on the switched controller design the authors refer to the case where switching can occur immediately. In real world there are many cases, where the switching signal rate of change is finite, that is $|\dot{\theta}| < \infty$. This assumption will be used in the proposed approach.

To separate the system matrix $A_{cl}(.)$ from the time derivative of the Lyapunov function $V(z, \theta, \xi)$ let us introduce the following two auxiliary matrices N_1, $N_2 \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ as follows

$$(2N_1 \dot{z} + 2N_2 z)^T (\dot{z} - A_{cl}(\theta, \xi) z) = 0$$

or after small manipulation

$$\begin{bmatrix} \dot{z}^T \\ z^T \end{bmatrix}^T \begin{bmatrix} N_1 + N_1^T & -N_1^T A_{cl}(\theta, \xi) + N_2 \\ \bullet & -N_2^T A_{cl}(\theta, \xi) - A_{cl}(\theta, \xi)^T N_2 \end{bmatrix} \begin{bmatrix} \dot{z} \\ z \end{bmatrix} = 0 \tag{12.11}$$

Assume that the positive definite Lyapunov function in (12.10) is in the form

$$V(z,\theta,\xi) = z^T P(\xi,\theta) z$$

The time derivative of the Lyapunov function is

$$\dot{V}(.) = \dot{z}^T P(\xi,\theta) z + z^T P(\xi,\theta)\dot{z} + z^T P(\xi,\dot{\theta}) z$$

or

$$\dot{V}(.) = \begin{bmatrix} \dot{z}^T & z^T \end{bmatrix} \begin{bmatrix} 0 & P(\xi,\theta) \\ P(\xi,\theta) & P(\xi,\dot{\theta}) \end{bmatrix} \begin{bmatrix} \dot{z} \\ z \end{bmatrix} \qquad (12.12)$$

where

$$P(\xi,\theta) = P_0(\xi) + \sum_{i=1}^{p} P_i(\xi)\theta_i, \quad P_i(\xi) = \sum_{j=1}^{N} P_{ij}\xi_j$$

$$P(\xi,\dot{\theta}) = \sum_{j=1}^{N}\sum_{i=1}^{p} P_{ij}\dot{\theta}_i\xi_j, \; j = 1,2,\ldots,N$$

Using(12.6) for performance (12.9) one obtains

$$J(t) = \begin{bmatrix} \dot{z}^T & z^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Q(\theta) + F_c^T R F_c \end{bmatrix} \begin{bmatrix} \dot{z} \\ z \end{bmatrix} \qquad (12.13)$$

Summing up (12.11), (12.12) and (12.13) and substitute the obtained result to (12.10), after small manipulations one obtains the robust stability condition for the gain-scheduled and switched controller design in the form

$$B_e = \begin{bmatrix} \dot{z}^T & z^T \end{bmatrix} L(\theta,\xi) \begin{bmatrix} \dot{z} \\ z \end{bmatrix} < 0 \qquad (12.14)$$

where

$$L(\theta,\xi) = \begin{bmatrix} l_{11}(\theta,\xi) & l_{12}(\theta,\xi) \\ l_{12}(\theta,\xi)^T & l_{22}(\theta,\xi)) \end{bmatrix} \qquad (12.15)$$

and

$$l_{11}(\theta,\xi) = N_1^T + N_1$$

$$l_{12}(\theta,\xi) = -N_1^T A_{cl}(\theta,\xi) + N_2 + P(\xi,\theta)$$

$$l_{22}(\theta,\xi) = Q(\theta) + F_c^T R F_c + P(\xi,\dot{\theta}) - N_2^T A_{cl}^T(\theta,\xi) - A_{cl}(\theta,\xi)^T N_2$$

$L(\theta,\xi)$ in (12.15) be affine with respect to vector variables θ and ξ. If $L(\theta,\xi) \leq 0$, the closed-loop system with the proposed robust gain-scheduled or switched controller is parameter dependent quadratically stable with guaranteed cost.

$L(\theta,\xi)$ can be rewritten as

$$L(\theta,\xi) = L_0(\xi) + \sum_{i=1}^{p} L_i(\xi)\theta_i \qquad (12.16)$$

where

$$L_0(\xi) = \begin{bmatrix} l_{011}(\theta,\xi) & l_{012}(\theta,\xi) \\ l_{12}(\theta,\xi)^T & l_{022}(\theta,\xi)) \end{bmatrix}$$

$$l_{011}(\xi) = N_1^T + N_1$$

$$l_{012}(\xi) = -N_1^T A_{0cl}(\xi) + N_2 + P_0(\xi)$$

$$l_{022}(\xi) = Q_0 + F_c^T R F_c + P(\xi,\dot{\theta}) - N_2^T A_{0cl}^T(\xi) - A_{0cl}(\xi)^T N_2$$

$$A_{0cl}(\xi) = \begin{bmatrix} A_0(\xi) + D_c C & B_0(\xi) C_c \\ B_{c0} C & A_{c0} \end{bmatrix}$$

$$L_i(\xi) = \begin{bmatrix} 0 & -N_1^T A_{icl}(\xi) + P_i(\xi) \\ * & -N_2^T A_{icl}(\xi) - A_{icl}(\xi)^T N_2 + Q_i \end{bmatrix}$$

$$A_{icl}(\xi) = \begin{bmatrix} A_i(\xi) + D_c C & B_i(\xi) C_c \\ B_{ci} C & A_{ci} \end{bmatrix}$$

Due to linearity $L(\theta,\xi)$ can be transformed to the form

$$L(\theta,\xi) = \sum_{j=1}^{N} (L_{0j} + \sum_{i=1}^{p} L_{ij}\theta_i)\xi_j = \sum_{j=1}^{N} M_j(\theta)\xi_j \tag{12.17}$$

where

$$L_{0j} = \begin{bmatrix} N_1^T + N_1 & -N_1^T A_{0clj} + N_2 + P_{0j} \\ * & Pe_0 + \sum_{i=1}^{p} P_{ij}\dot{\theta}_i - N_2^T A_{0clj} - A_{0clj}^T N_2 \end{bmatrix}$$

$$Pe_0 = Q_0 + F_c^T R F_c$$

$$A_{0clj} = \begin{bmatrix} A_{0j} + B_{0j} D_c C & B_{0j} C_c \\ B_{c0} C & A_{c0} \end{bmatrix}$$

$$L_{ij} = \begin{bmatrix} 0 & -N_1^T A_{iclj} + P_{ij} \\ * & Q_i - N_2^T A_{iclj} - A_{iclj}^T N_2 \end{bmatrix}$$

$$A_{iclj} = \begin{bmatrix} A_{ij} + B_{ij} D_c C & B_{ij} C_c \\ B_{ci} C & A_{ci} \end{bmatrix}$$

$$M_j(\theta) = L_{0j} + \sum_{i=1}^{p} L_{ij}\theta_i, \quad j = 1,2,\ldots,N \tag{12.18}$$

Since (12.16) and (12.17) are convex with respect to parameters θ and ξ, the above inequalities split to N inequalities of (12.18). The obtained sufficient robust stability conditions are summarized in the following theorem.

Theorem 12.1. *LPV uncertain continuous-time system (12.1) with Gain-Scheduled or Switched (with arbitrarily switching algorithm) dynamic output feedback controller (12.6) is parameter dependent quadratically stable with guaranteed cost (12.9) if there exist matrices N_1, N_2 and control algorithm (12.6) such that the following inequality hold*

$$M_j(\theta) < 0, \; j = 1,2,\ldots,N \tag{12.19}$$

that is if in all vertices of $j = 1,2,\ldots,N$ and $i = 1,2,\ldots,p$ $M_j(\theta)$ has a negative value.

Notes:

- For the case of ideal switching the rate of the switching algorithm is infinite ($\dot{\theta}_i \to \infty(-\infty)$). For this case in (12.12) the matrix $P_i(\xi) = 0$.
- For the case of non-ideal switching algorithm $|\dot{\theta}_i| < \infty$ the switching controller design procedure and the gain-scheduled controller design procedure are the same.
- For the case of quadratic stability approach with respect to uncertainty in (12.12) one puts $P(\xi, \theta) = P_0 + \sum_{i=1}^{p} P_i \theta_i$.

12.4 Examples

12.4.1 Prescribed controller structure

Assume that instead of controller (12.7) one wants to use the standard PID or PI gain-scheduled or switched controller with transfer function

$$R(s) = \frac{b_{n_c-1}s^{n_c-1} + \dots b_0}{s_c^n + a_{n_c-1}s^{n_c-1} + \dots a_0}$$

In this case the structure of matrices A_{ci}, B_{ci}, C_c, $D_c = 0$, $i = 0, 1, \dots, p$ needs to be prescribed. For SISO case the controller prescribed matrices are in the form

$$A_{ci} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ .. & .. & .. & .. & .. \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}; B_{ci} = \begin{bmatrix} b_0 \\ b_1 \\ .. \\ b_{n-1} \end{bmatrix}, i = 0, 1, \dots, p \tag{12.20}$$

$$C_c = [0 \dots 0 \quad 1]$$

For PID controller with filter one obtains

$$R(s) = \frac{k_p}{T_f s + 1}\left(1 + \frac{1}{T_i s} + \frac{T_d s}{T_d/N_c s + 1}\right) \tag{12.21}$$

$$a_0 = 0; \; a_1 = \frac{N_c}{T_f T_d}; \; a_2 = \frac{N_c T_f + T_d}{T_d T_f}$$

$$b_0 = \frac{N_c k_p}{T_i T_d T_f}; \; b_1 = \frac{N_c T_i + T_d}{T_i T_d T_f} k_p; \; b_2 = \frac{N_c + 1}{T_d T_f} k_p$$

If one chooses T_f and N_c, the other controller parameters from (12.21) are calculated straightforward. For PI controller with transfer function

$$R(s) = \frac{k_p}{T_f s + 1}\left(1 + \frac{1}{T_i s}\right) \tag{12.22}$$

the matrices in (12.20) are

$$a1 = \frac{1}{T_f}; \; b_0 = \frac{k_p}{T_i T_f}; \; b_1 = \frac{k_p}{T_f}$$

When $a_0 = 0$, a_1, b_0, b_1 are known, one obtains simple equations for PI controller parameter calculation

$$T_f = \frac{1}{a_1};\ k_p = b_1 T_f;\ T_i = \frac{k_p}{b_0 T_f} \tag{12.23}$$

12.4.2 Different quadratic stability approach

In the proposed example, the results obtained using parameter dependent quadratic stability are compared with the results for different quadratic stability variants. The following 4 variants of the Lyapunov function are used in the design procedure to study the differences between the qualities of the designed controllers.

- *DP1*: Quadratic stability with respect to uncertain model parameter variation. For this case, the Lyapunov matrix is dependent only on θ and it is in the form

$$P(\theta) = P_0 + \sum_{i=1}^{s} P_i \theta_i \tag{12.24}$$

- *DP2*: Parameter dependent quadratic stability. The Lyapunov matrix depends on both ξ and θ and is given as

$$P(\xi, \theta) = P_0(\xi) + \sum_{i=1}^{s} P_i(\xi)\theta_i \tag{12.25}$$

  where

$$P_j(\xi) = \sum_{i=1}^{N} P_{ji}\xi_i, \quad j = 0, 1, 2, \ldots, s, \quad \sum_{i=1}^{N} \xi_i = 1$$

- *DP3*: Quadratic stability with respect to gain-scheduled parameters. For this case the Lyapunov matrix is dependent only on ξ

$$P(\xi) = \sum_{i=1}^{N} P_i \xi_i \tag{12.26}$$

- *DP4*: Quadratic stability with respect to both gain-scheduled and uncertain parameters. The Lyapunov matrix is P_0, independent of ξ and θ.

12.4.3 Robust controller design

The uncertain gain-scheduled plant model (12.1) for the case of $p = 1$, $N = 4$ is given as follows:

$$A_{01} = \begin{bmatrix} -0.5 & 0.2 & 0.3 \\ .2 & -0.78 & 1 \\ 0.05 & -0.02 & -.1 \end{bmatrix}, A_{02} = \begin{bmatrix} -0.3 & 0.15 & 0.25 \\ .25 & -0.5 & .7 \\ 0.05 & 0.01 & -.15 \end{bmatrix}$$

$$A_{03} = \begin{bmatrix} -0.6 & 0.15 & 0.15 \\ .4 & -0.8 & .75 \\ 0.01 & 0.06 & -.5 \end{bmatrix}, \quad A_{04} = \begin{bmatrix} -0.35 & 0.25 & 0.17 \\ .18 & -0.22 & .23 \\ 0.02 & -0.043 & -.3 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} -0.65 & 0.15 & 15 \\ .4 & -0.8 & 0.75 \\ 0.015 & 0.06 & -.5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.38 & 0.215 & 0.25 \\ .25 & -0.5 & .77 \\ 0.05 & 0.01 & -.15 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} -0.33 & 0.15 & 0.25 \\ .25 & -0.51 & .705 \\ 0.105 & 0.15 & -.21 \end{bmatrix}, A_{14} = \begin{bmatrix} -0.35 & 0.25 & 0.17 \\ .18 & -0.22 & .23 \\ 0.02 & -0.043 & -.3 \end{bmatrix}$$

$$B_{01} = \begin{bmatrix} .01 \\ 0.5 \\ 0.08 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} .03 \\ 3 \\ 0.23 \end{bmatrix}$$

$$B_{03} = \begin{bmatrix} .015 \\ .55 \\ 0.08 \end{bmatrix}, \quad B_{04} = \begin{bmatrix} .01 \\ .52 \\ 0.088 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} .015 \\ 0.52 \\ 0.088 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} .015 \\ 1 \\ 0.2 \end{bmatrix}$$

$$B_{13} = \begin{bmatrix} .01 \\ 2 \\ 0.3 \end{bmatrix}, \quad B_{14} = \begin{bmatrix} .02 \\ 1.5 \\ 0.3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

The problem is to design to the linear uncertain system (12.1) robust gain-scheduled controller given by (12.6) which will ensure the closed-loop stability (12.8), guaranteed cost and parameter dependent quadratic stability for the case of the following parameters:

- the gain-scheduled parameters $\theta_1 \in \langle -0.15, 0.15 \rangle$, $p = 1$, $\dot{\theta}_1 \in \langle -5, 5 \rangle$, for the Lyapunov matrix $0 < P(\xi, \theta) = P_0(\xi) + \sum_{i=1}^{p} P_i(\xi)\theta_i < roI$, $ro = 1000$ holds. The parameters of performance (12.9) are as follows: $Q_0 = 0.02 * I$, $Q_1 = 0.002 * I$, $R = I$.

- the gain-scheduled parameters $\theta_1 \in \langle 0, 1 \rangle$, $p = 1$, $\dot{\theta}_i \in \langle -100, 100 \rangle$. Performance specification $Q_0 = 0.02 * I$, $Q_1 = 0.002 * I$, $R = I$ and Lyapunov matrix limitation : $0 < P(\xi, \theta) = P_0(\xi) + \sum_{i=1}^{p} P_i(\xi)\theta_i < roI$, $ro = 1000$. Due to the value of $\dot{\theta}_i$ for the second case the obtained gain-scheduled controller can work in the regime of switched controller with arbitrary switching.

- Note that for the design of the switched controller $p \geq 2$.

- To obtain a feasible solution in the unified gain-scheduled and switched controller design procedure one can use a free and open source of BMI solver Penlab.

The first case of robust gain-scheduled controller design

For the case of a prescribed gain-scheduled dynamic output feedback controller structure and the above different quadratic stability approaches the obtained robust gain-scheduled controller parameters are given below as follows:

1. Case DP1: prescribed gain-scheduled dynamic output feedback controller structure

$$
\begin{array}{llll}
A_{C0} = & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2.2189 \\ 0 & 1 & -1.7354 \end{bmatrix} & B_{C0} = & \begin{bmatrix} -0.0528 \\ -1.2598 \\ -1.8443 \end{bmatrix} \\
A_{C1} = & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1.856 \\ 0 & 1 & -1.8287 \end{bmatrix} & B_{C1} = & \begin{bmatrix} -0.117 \\ -1.5733 \\ -1.9692 \end{bmatrix} \\
C_C & = [0\,0\,1] & D_C & = [0]
\end{array}
$$

In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.0126$.

2. Case DP2: prescribed gain-scheduled dynamic output feedback controller structure

$$
\begin{array}{llll}
A_{C0} = & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1.7120 \\ 0 & 1 & -1.499 \end{bmatrix} & B_{C0} = & \begin{bmatrix} -0.1004 \\ -1.2506 \\ -1.1308 \end{bmatrix} \\
A_{C1} = & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1.5976 \\ 0 & 1 & -1.6212 \end{bmatrix} & B_{C1} = & \begin{bmatrix} -0.1955 \\ -1.0696 \\ -1.1819 \end{bmatrix} \\
C_C & = [0\,0\,1] & D_C & = [0]
\end{array}
$$

In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.0304$.

3. Case DP3: gain-scheduled dynamic output feedback controller, general structure

$$
\begin{array}{llll}
A_{C0} = & \begin{bmatrix} -6.1307 & 0 & 0 \\ 0 & -6.1307 & 0 \\ 0 & 0 & -6.4417 \end{bmatrix} & B_{C0} = & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
A_{C1} = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0530 \end{bmatrix} & B_{C1} = & \begin{bmatrix} 0 \\ 0 \\ -0.644 \end{bmatrix} \\
C_C & = [0\,0\,1] & D_C & = [-0.6676]
\end{array}
$$

In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.1486$.

4. Case DP4: gain-scheduled dynamic output feedback controller, general structure

$$
\begin{array}{llll}
A_{C0} = & \begin{bmatrix} -5.6621 & 0 & 0 \\ 0 & -5.6621 & 0 \\ 0 & 0 & -31527 \end{bmatrix} & B_{C0} = & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
A_{C1} = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.8794 \end{bmatrix} & B_{C1} = & \begin{bmatrix} 0 \\ 0 \\ 0.2589 \end{bmatrix} \\
C_C & = [0\,0\,1] & D_C & = [-0.6471]
\end{array}
$$

In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.1165$.

The second case of robust gain-scheduled-switched controller design
For the case of different prescribed gain-scheduled-switched dynamic output feedback controller structure and the above different quadratic stability approaches the obtained robust gain-scheduled controller parameters are given below as follows:

1. Case DP1: prescribed gain-scheduled-switched controller structure

$$\begin{aligned} A_{C0} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -23202 \\ 0 & 1 & -1.3286 \end{bmatrix} & B_{C0} &= \begin{bmatrix} -0.0916 \\ -1.3158 \\ -10984 \end{bmatrix} \\ A_{C1} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1.83232 \\ 0 & 1 & -2.0471 \end{bmatrix} & B_{C1} &= \begin{bmatrix} -0.1782 \\ -1.2271 \\ -1.5157 \end{bmatrix} \\ C_C &= [0\ 0\ 1] & D_C &= [0] \end{aligned}$$

   In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.0215$.

2. Case DP2: prescribed gain-scheduled-switched controller structure

$$\begin{aligned} A_{C0} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1.9749 \\ 0 & 1 & -11883 \end{bmatrix} & B_{C0} &= \begin{bmatrix} -0.1086 \\ -1.1485 \\ -0.7984 \end{bmatrix} \\ A_{C1} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1.7053 \\ 0 & 1 & -1.5816 \end{bmatrix} & B_{C1} &= \begin{bmatrix} -0.1592 \\ -0.8254 \\ -0.9119 \end{bmatrix} \\ C_C &= [0\ 0\ 1] & D_C &= [0] \end{aligned}$$

   In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.0307$.

3. Case DP3: gain-scheduled-switched dynamic output feedback controller

$$\begin{aligned} A_{C0} &= \begin{bmatrix} -6.4835 & 0 & 0 \\ 0 & -6.4835 & 0 \\ 0 & 0 & -6.1245 \end{bmatrix} & B_{C0} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ A_{C1} &= \begin{bmatrix} -0.3309 & 0 & 0 \\ 0 & -0.3309 & 0 \\ 0 & 0 & -1.3723 \end{bmatrix} & B_{C1} &= \begin{bmatrix} 0 \\ 0 \\ 0.0509 \end{bmatrix} \\ C_C &= [0\ 0\ 1] & D_C &= [-0.5283] \end{aligned}$$

   In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.1240$.

4. Case DP4: gain-scheduled-switched dynamic output feedback controller

$$\begin{aligned} A_{C0} &= \begin{bmatrix} -8.1281 & 0 & 0 \\ 0 & -8.1281 & 0 \\ 0 & 0 & -8.2784 \end{bmatrix} & B_{C0} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ A_{C1} &= \begin{bmatrix} -0.0411 & 0 & 0 \\ 0 & -0.0411 & 0 \\ 0 & 0 & -1.0924 \end{bmatrix} & B_{C1} &= \begin{bmatrix} 0 \\ 0 \\ 0.1070 \end{bmatrix} \\ C_C &= [0\ 0\ 1] & D_C &= [-0.4338] \end{aligned}$$

   In the polytope vertices closed loop system the maximal eigenvalue value for the case of $\theta_1 = 0$ is $Maxeig = -0.1153$.

Simulation results (*Figs.* 12.1, 12.2, 12.3, 12.4) confirm, that *Theorem* holds. Simulation results for the first case (robust gain-scheduled controller design) are shown in *Figs.* 12.1 and 12.2, where the scheduled parameter is calculated from the system output. Simulation results for the second case (robust gain-scheduled-switched controller design)

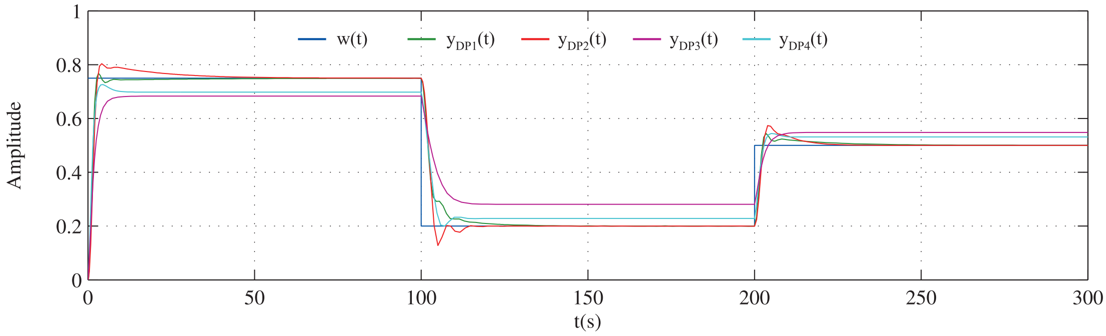

FIGURE 12.1: Simulation results $y(t)$, $w(t)$ for the first case of robust gain-scheduled controllers

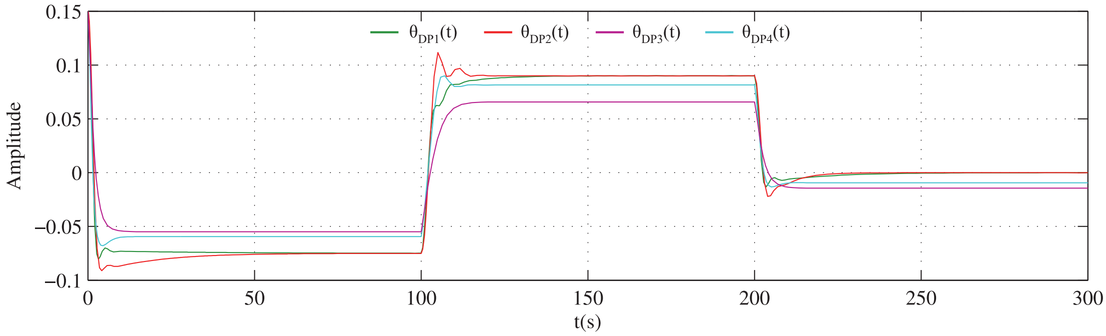

FIGURE 12.2: Calculated scheduled parameters $\theta_{DP1-4}(t)$ for the first case of robust gain-scheduled controllers

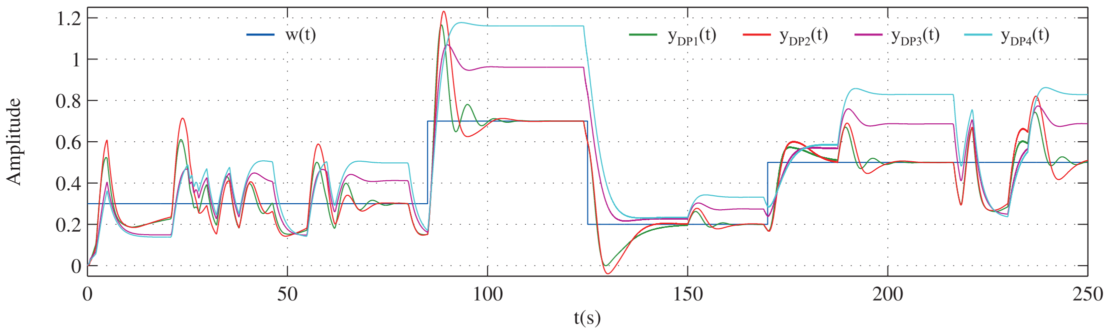

FIGURE 12.3: Simulation results $y(t)$, $w(t)$ for the second case of robust gain-scheduled controllers

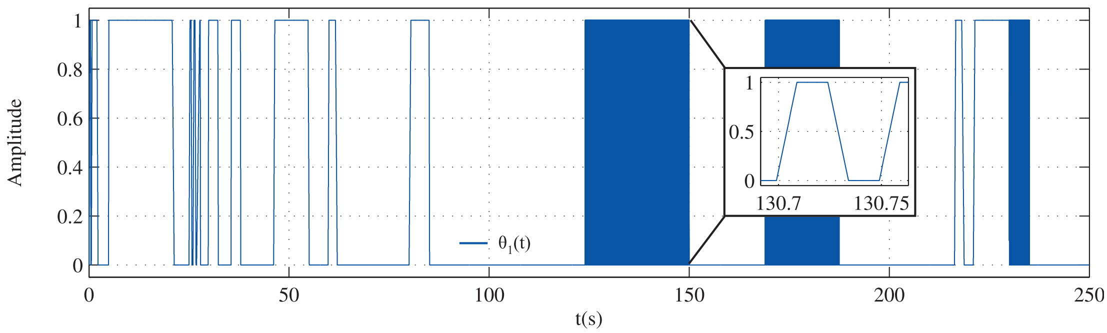

FIGURE 12.4: Scheduled parameter $\theta_1(t)$ for the second case of robust gain-scheduled controllers

are shown in *Figs.* 12.3 and 12.4. The scheduled parameters from a randomly generated switching algorithm are shown in *Fig.* 12.4, where the maximal rate of change is $\dot{\theta} = 100$ $1/s$.

12.5 Conclusion

The proposed paper addresses the problem how to obtain the new unified procedure to design a robust gain-scheduled and switched controller with arbitrary switching for continuous-time systems described by a novel robust plant model using the parameter dependent quadratic stability (PDQS) approach. The obtained unified controller design procedure ensures the closed-loop stability and guaranteed cost for a prescribed rate of change of the system switching (gain-scheduled) variable. In some real cases the rate of change of the switching signal is finite. This assumption was used in the paper to obtain the switched controller design procedure. The advantages of the proposed method are:

- one can obtain less conservative results in comparison with using the dwell-time approach,
- for the switched controller design there is no need to use the approach of "dwell-time" markedly complicating the design procedure,
- the rate of the switching signal (scheduled variables) change can be prescribed by the designer, which opens the new possibilities for practical realizations and development of new theoretical approaches,
- the obtained design procedure for output/state feedback ensures the closed loop robust stability of gain-scheduled or switched systems and guaranteed cost,
- the obtained design procedure can be implemented easily to the standard LMI or BMI approaches.

Numerical examples illustrate the effectiveness of the proposed approach.

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Concluding remarks

13.1 Brief overview

This thesis deals with controller design for nonlinear systems. The controller is given in a feedback structure that is, the controller has information about the system and uses it to influence the system. The nonlinear system is transformed to a linear parameter-varying system which is used for controller design, i.e. gain-scheduled controller design. The gain-scheduled controller synthesis presented in this thesis is based on the Lyapunov theory of stability as well as on the Bellman-Lyapunov function. To achieve a performance quality, a quadratic cost function and its modifications known from LQ theory are used. The obtained gain-scheduled controller guarantees the closed-loop stability and the guaranteed cost. The main results for controller synthesis are in the form of bilinear matrix inequalities and/or linear matrix inequalities. For controller synthesis, one can use a free and open source BMI solver PenLab or LMI solvers LMILab, SDPT3 or SeDuMi.

13.2 Closing remarks and future works

The main goal for this thesis (and also to our research in last 2,5 – 3 years) was to find a systematic controller design approach for uncertain nonlinear systems, which guarantees the closed-loop stability and guaranteed cost with considering input/output constraints, all this without on-line optimization and need of high-performance industrial computers. We tried to select those publications which most closely reflect the achieved results. The first included paper (*Chapter 4*) presents a simple gain-scheduled controller design for nonlinear systems which guarantees the closed-loop stability and guaranteed cost. One can include the maximal value of the rate of gain-scheduled parameter changes which allows obtaining the controller with a given performance and decreased conservativeness. In the next chapter (*Chapter 5*), one can find a simple modification of these results, where a new quadratic cost function is used, where weighting matrices are time-varying and depending on scheduled parameter. Using these original variable weighting matrices, we can affect performance quality separately in each working point and we can tune the system to the desired condition through all parameter changes. *Chapters 6, 7* present the robust versions of the obtained results from *Chapters 4, 5*, where in *Chapter 6* the design procedure is transformed from the bilinear matrix inequality form to linear matrix inequality, which caused that our controller synthesis works for high order systems. In *Chapter 8*, a simplified version of the robust controller design in discrete time domain is presented, where a new LPV description of T1DM Bergman's minimal model with two additional subsystems (absorption of digested carbohydrates and subcutaneous insulin absorption) is created. The controller synthesis in this paper is also transformed to LMI problem. In *Chapters 9* and *10*, gain-scheduled controller designs adopted to switched control are presented in continuous time. In the proposed design procedures there is no need to use the notion of the "dwell-time" for arbitrary switching, which significantly simplifies the switched controller design compared to approaches in the literatures. In *Chapter 11*, a novel gain scheduling based model predictive controller design procedure for nonlinear systems is presented for finite and infinite prediction horizons with considering input/output constraints. Finally, a novel unified robust gain-scheduled and switched controller design approach is presented in *Chapter 12*, where the conservativeness from multi-convexity is eliminated.

The stated objectives (in *Chapter 1*) were reached successfully but there are still many unsolved problems. For example, in this thesis it is hypothesized that the scheduled parameters can be measured and the measurement is accurate. It is true that if one uses the robust version, then the measurement inaccuracy can be covered as model uncertainty but this should be studied in more detail. Furthermore, it would be good to study how information from disturbances can be used to improve the performance quality under disturbances. Moreover, it would be an interesting study how to reduce the time required to controller synthesis because it is well known that the time required for controller design using LMI and especially BMI solvers rapidly increases for higher order systems. It follows that this thesis opens new possibilities for further studies and research in this specific area.

Appendix A

List of publications

Original research papers in international scientific journals (peer- reviewed)

[1] VESELÝ, Vojtech – ILKA, Adrian. Gain-Scheduled PID Controller Design. Journal of Process Control, 2013, vol. 23, p. 1141-1148.

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